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DYNAMICS,

OR

AN ELEMENTARY

TREATISE ON MOTION;

WITH

A GREAT VARIETY OF EXAMPLES

ILLUSTRATIVE OF THE GENERAL PRINCIPLES AND FORMULÆ:

TO WHICH IS ADDED,

A SHORT

TREATISE ON ATTRACTIONS.

By S. EARNSHAW, B.A.,

Of St. John's College, Cambridge.

CAMBRIDGE:
PRINTED BY W. METCALFE, ST. MARY'S STREET,
FOR J. & J. J. DEIGHTON;
AND WHITTAKER, TREACHER & ARNOT, AVE-MARIA-LANE, LONDON.

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PREFACE.

In the following Treatise on Dynamics, it has been the principal aim of the Author to set the subject before the reader in a clear and concise manner. For this reason, he has not interrupted the steps of the demonstrations by introducing proofs of Formulæ or Principles belonging to other subjects, with which the reader might be supposed to be already acquainted. As, however, some of the Formulæ, depending on the Differential Calculus, which occur in the work, are not in very common use, a list of the most unusual is subjoined, which the reader will find particularly useful in matters relating to Central Forces. The notation is explained in Art. 87.

Twice the differential coefficient of a Spiral Area, with regard to *any* quantity t, of which it is a function,

$$= y d_t x - x d_t y,$$

$$= p d_t s,$$

$$= r^2 d_t \theta.$$

Also, $\frac{1}{p^2} = u^2 + d_\theta u^2.$

Also, the Radius of Curvature

$$
\begin{aligned}
&= \frac{d_t s^3}{d_t y\, d_t^2 x - d_t x\, d_t^2 y}, \\
&= \frac{1}{d_s y\, d_s^2 x - d_s x\, d_s^2 y}, \\
&= \frac{1}{\sqrt{(d_s^2 x)^2 + (d_s^2 y)^2}}, \\
&= \frac{d_s y}{d_s^2 x}, \\
&= -\frac{d_s x}{d_s^2 y}, \\
&= -\frac{d_x s^3}{d_x^2 y}, \\
&= r d_p r.
\end{aligned}
$$

Lastly, the Chord of Curvature passing through the origin

$$
\begin{aligned}
&= 2 p d_p r, \\
&= -2 d_s x . \frac{d_x s^3}{d_x^2 y}, \\
&= -\frac{2 d_x s^2}{d_x^2 y}.
\end{aligned}
$$

As this volume has been written principally for the use of Students in the University, the demonstrations of many things are given at length, which might have been deduced at once from propositions previously proved.

CAMBRIDGE,
1st. *May*, 1832.

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DYNAMICS.

CHAPTER I.

DEFINITIONS AND FIRST PRINCIPLES.

1. *Dynamics* is that part of MECHANICS which treats of Motion.

2. When a particle of matter is continually changing its position in space, it is said to be *in motion:* and the line which passes through all the successive positions it has occupied during its motion, is called its *path.*

3. Whatever causes, or tends to cause, motion in a body, is called *force.*

Forces are divided into several kinds; the division being founded upon some particular circumstances under which the force communicates the motion. For example,—

Pressure is the force which a body in motion, or having a tendency to motion, exerts on an obstacle, or other body, with which it is in contact.

Impact is a sudden pressure, exerted for a very short, though finite, time.

Attraction is the name given to the force which a body is supposed to exert on other bodies, which are observed to have a continual tendency towards it, although there are no discoverable connecting parts between it and them.

4. A body is said to be *free,* when there is no impediment to its motion in any direction whatsoever.

5. If a force act upon a free particle at rest, the motion must, of necessity, take place in the direction in which the force acts.

For if not, there is the same reason that the particle will move to the right hand as to the left of that direction, and therefore it will move in two directions at once: which is absurd.

6. But although we are able to determine the direction in which a force produces its effect, yet we know so little of the nature of forces in general, that we are compelled to estimate their magnitudes by the effects produced: and hence, if the effects separately produced on a body by two forces in equal times are observed to be equal, we say that the forces themselves are equal, without paying any regard to their nature.

7. Motion is said to be *uniform,* when the space described in a given interval is equal to that described in any other equal interval. It is said to be *variable* when these spaces are unequal.

Bodies, with different degrees of uniform motion, would in a given interval of time describe paths of different lengths; on this account the motion of a body is estimated by its velocity.

8. The *Velocity* of a body at a given point of its path, is the space it would describe in a unit of time, if the motion were at that point to become uniform.

The unit of time is generally *one second,* and the length of the path is usually expressed in *feet.*

Hence, in uniform motion, if s be the space described in the time t seconds, with velocity v,

$$s = v . \frac{t''}{1''} = v t,$$

in which expression t does not denote length of time, but an abstract number or ratio.

9. Cor. If the space described in any time $\propto$ as that time, the motion is uniform, and the vel. $= \frac{space}{time}$.

10. Motion is said to be *accelerated* when the velocity continually *increases,* and *retarded* when it continually *decreases.*

THE LAW OF INERTIA.

11. *A particle of matter cannot, by any action of its own upon itself, produce a change in its present state of rest or motion.*

First. If a particle be free, acted on by no external force, and at rest, it will continue at rest.

As far as motion is concerned, the particle is exactly in the same case as if all other matter in the universe were annihilated; it is therefore similarly related to every part of space; consequently, if it begin to move at all, it must move in every direction at the same instant, for there is no reason why it should move in one direction rather than another: which is absurd. Therefore it will remain at rest.

Secondly. Under the same circumstances if it be in motion, it will continue to move uniformly in a straight line.

For having no external relation, there is as much reason for its deviating to the right hand as to the left of its present direction, and therefore it will not deviate at all. Again, we can retard its motion by the application of a force in the contrary direction; that is, by establishing a certain *relation* between the particle and that part of space *from* which it is moving; and as we cannot judge of forces otherwise than by their effects, we conclude, whenever we observe the motion to be retarded, that some relation (of the nature of force) exists between the particle and that part of space from which it is moving, which (relation) does not exist between it and any other part. Now it seems to be absurd to suppose that matter can establish any relation whatever between itself and any one part of space in preference to another, which it must do if it retard its own motion. Therefore the motion will be uniform.

12. Cor. 1. Motion, and change of motion, can only be produced by the action of force.

13. Cor. 2. and Def. If the velocity of a particle continually *increases* or *decreases*, it is acted on respectively by an *accelerating* or *retarding* force; which is said to be *uniform* or *variable*, according as the increments or decrements of velocity in equal times, are *equal* or *unequal*.

14. By the *Law of Inertia*, no part of velocity generated is destroyed or lost by the particle, and therefore we may *measure* an accelerating force by the velocity generated by it, (supposing it to become uniform) in a unit of time ($1''$).

15. The *momentum* or quantity of motion of a body, is the sum of the motions of all its particles; and as the motion of a particle is measured by its velocity (Art. 7), the momentum will be measured by the velocity into the number of particles, and will $\therefore =$ (mass).(vel.), supposing all the particles to move with the same velocity. In other cases the momentum of the particle $m = mv$, and $\therefore$ momentum of the body $= \Sigma\,(mv)$.

16. *Moving force* is the momentum generated in a unit of time ($1''$); and $\therefore =$ (mass).(accelerating force) in the former case; and $= \Sigma\,(mf)$ in the latter; where $f =$ accelerating force on m.

THE LAW OF THE COMPOSITION OF FORCES.

17. *If a particle of matter be acted on by any number of forces, each one will produce its* DUE *effect, estimated in the direction in which it acts.*

By the effect *due* to a force, is here meant, that which it would produce upon a free, quiescent particle.

This Law may be proved by such experiments as the following:—

A pendulum will vibrate, at a given place, in the same time from east to west, north to south, or in any other direction. A body thrown up an inclined plane with a given velocity, will describe the same path on the plane to what point soever of the compass it be inclined. We conclude from these two experiments that the motion of the Earth in its orbit and round its axis, produces no alteration in the effect of gravity in the one case, and the force of projection in the other.

Again, a body let fall from the top of a ship's mast will fall at the foot of it, in the same time; whether the ship be at anchor, or sailing uniformly. This experiment shews that the body when let fall retains the horizontal motion it previously had in common with the ship, and that gravity still produces its due effect in the direction in which it acts.

18. Cor. 1. If a force always act upon a body with the same intensity, the velocity generated in any time will, by this law, be independent of that which it previously had; and therefore the force will generate equal velocities in equal times. Hence a uniform force is one which always acts with the same intensity.

19. Cor. 2. For similar reasons, two equal forces will produce double the effect of one; three, treble; &c. Hence the velocity generated in a *given* mass in a given time is proportional to the force impressed.

20. Cor. 3. Since the velocity generated $\propto$ force impressed, we may apply all the rules laid down in Statics for the composition and resolution of forces, to the composition and resolution of velocities and uniform motions.

Hence if v_1 v_2 v_3 be the velocities of a particle $\parallel$ the rectangular co-ordinate axes of x y z, the actual velocity $=\sqrt{v_1^2 + v_2^2 + v_3^2}$.

21. Cor. 4. If the equal forces F_1, F_2, F_3, . . . act respectively upon the equal free particles m_1, m_2, m_3, ... the velocity v generated in each will be the same, and will continue unaltered, if we suppose them to coalesce during the motion so as to form one body: hence that the velocity generated in different bodies in a given time may be the same, the force acting on each must be proportional to the mass (m); that is, when v is given, force $\propto m$; but (Cor. 3) when m is given force $\propto v$, and $\therefore$ in general, *force* $\propto$ mv $\propto$ *momentum communicated* in a given time.

22. Cor. 5. The moving force is proportional to the impressed force.

23. Cor. 6. Since a motion cannot be communicated to matter without the action of force; it follows, that matter offers a certain *resistance* (called its *inertia*, *vis inertiæ*, and in some circumstances its *reaction*) to the communication of motion, which is *equal*, and exerted in a direction *opposite*, to the force required to overcome it, (which force has been called *action* by some writers): this is all that is meant by saying that *action and reaction are equal and opposite.*

24. Cor. 7. Hence when one body exerts an influence upon another by *attraction*, *pressure*, *impulse*, &c., the momentum destroyed by the reaction will be equal to that communicated by the action. (Art. 21.)

25. It has been usual amongst English writers to demonstrate the Law of Inertia by experiment. But that this method

is not correct will be evident, if we consider that we are not acquainted with a single particle of matter in the whole universe that can be said, even approximately, to be at rest, or that is not acted upon by an almost infinite variety of forces; and that we ourselves, placed upon a planet, and constantly undergoing with it many complicated motions, are as unable to make any experiment upon a body to prove that Law, as we are to find a particle at rest or acted on by no forces on which to make the experiment.

If the subject be closely examined, it will be found that all the experiments which have been brought forward to prove the Law of Inertia, belong in reality to the proof of the Law of the Composition of Forces.

CHAPTER II.

ON IMPACT.

26. In this chapter impinging bodies are always supposed to be spherical and of uniform density.

The impact of two bodies is said to be *direct,* when their centres always move in the straight line passing through the point of impact. In other cases it is said to be *oblique.*

It is found, by experiment, that most bodies suffer compression during impact, and recover their original figure, after compression, with a force, which, in the same material, is to that by which they were compressed in a *constant ratio.*

The force which a body exerts to recover its former figure, is called its *elasticity;* but sometimes the fraction $\frac{\textit{force of elasticity}}{\textit{force of compression}}$ is called its *elasticity.*

The body is said to be *perfectly elastic* when force of elasticity = force of compression.

Hardness consists in a firm cohesion of the particles of a body, so as to render it difficult to change their relative situations; and when no finite force can effect a change, the body is said to be *perfectly hard.*

It is shewn in STATICS that to find the velocity of a body in a given direction, we must constantly draw perpendiculars from the body upon that direction, and the velocity required will be that of the foot of the perpendicular.

The *relative velocity* of two bodies in a given direction, is that velocity with which the perpendiculars just mentioned approach towards, or recede from, each other.

27. In the direct impact of two inelastic bodies, to find their velocity after impact.

Let A, B be the masses of the two bodies; a, b their respective velocities reckoned in the direction of A's motion. Then $Aa + Bb$ is their momentum before impact; and during impact the momentum lost by one will be gained by the other (Art. 24), and ∴ the whole momentum will be unaltered; and the bodies will move on together after impact, because there is no elastic force to separate them.

$$\therefore (A + B).\text{ common velocity} = Aa + Bb.$$

$$\therefore \text{velocity after impact} = \frac{Aa + Bb}{A + B}.$$

Cor. This is also the velocity of their common centre of gravity before impact, and ∴ the velocity of the centre of gravity of two bodies is unaltered by direct impact.

28. If λ denote the common elasticity of two bodies A, B; then in direct impact, $A + B : (1 + \lambda) A ::$ relative velocity before impact : velocity gained by B, and $A + B : (1 + \lambda) B ::$ relative velocity before impact : velocity lost by A; estimating the velocities in the direction of A's motion.

For during the compression A accelerates B's motion, and B retards A's until their velocities are equal, at which instant the compression ceases, and their common velocity $= \frac{Aa+Bb}{A+B}$, and ∴ velocity lost by $A = a - \frac{Aa+Bb}{A+B} = \frac{B(a-b)}{A+B}$, and that gained by $B = \frac{Aa + Bb}{A + B} - b = \frac{A(a - b)}{A + B}$. But now the elasticity begins to act, and produces effects which are to those produced by compression $:: \lambda : 1$, (Art. 26); ∴ the whole velocity lost by $A = \frac{(1 + \lambda) B (a - b)}{A + B}$, and that gained by $B = \frac{(1 + \lambda) A (a - b)}{A + B}$.

29. Cor. 1. The relative velocity after impact : relative velocity before impact :: $\lambda : 1$.

For relat. vel. after impact = B's vel. — A's vel.

$$= b - a + \frac{(1+\lambda)A(a-b)}{A+B} + \frac{(1+\lambda)B(a-b)}{A+B}$$

$$= \lambda(a-b).$$

30. Cor. 2. If A impinge on B at rest, it will lose its whole velocity if $\frac{(1+\lambda)Ba}{A+B} = a$, or, when A : B :: $\lambda : 1$.

31. In the direct impact of two perfectly elastic bodies, the sum of their *vires vivæ* is the same before and after impact.

Def. The *vis viva* of a body is equal the product of its mass into the square of its velocity.

Let α, β be their respective velocities after impact; then, as in Art. 27, the whole momentum is unaltered by impact;

$$\therefore Aa + Bb = A\alpha + B\beta,$$

$$\text{or, } A(a-\alpha) = B(\beta - b);$$

$$\text{but } a + \alpha = \beta + b, \text{ (Art. 29).}$$

$a - b = \beta - \alpha$

$\therefore$ multiplying both equations together,

$$A(a^2 - \alpha^2) = B(\beta^2 - b^2),$$

$$\text{or, } Aa^2 + Bb^2 = A\alpha^2 + B\beta^2.$$

32. Cor. 1. The proposition is true whatever be the number of bodies, providing they are all perfectly elastic, and A impinge on B, B on C, &c. . . .

$$\therefore Aa^2 + Bb^2 + Cc^2 + \&c. = A\alpha^2 + B\beta^2 + C\gamma^2 + \ldots.$$

$$\text{or, } \Sigma(Aa^2) = \Sigma(A\alpha^2).$$

33. Cor. 2. The proposition does not hold good when the bodies are imperfectly elastic.

For $\lambda = \frac{\beta - \alpha}{a - b}$ (Art. 29); $\therefore \frac{1-\lambda}{1+\lambda} = \frac{a + \alpha - \beta - b}{a - \alpha + \beta - b}$,

$$\therefore\ a + \alpha - \frac{1-\lambda}{1+\lambda}(a-\alpha) = \beta + b + \frac{1-\lambda}{1+\lambda}(\beta - b),$$

and multiplying this equation by $A(a-\alpha) = B(\beta - b)$ and transposing, we have

$$Aa^2 + Bb^2 = A\alpha^2 + B\beta^2 + \frac{1-\lambda}{1+\lambda}\{A(a-\alpha)^2 + B(b-\beta)^2\}.$$

which coincides with the result of Art. 31, only when $\lambda = 1$, or when the bodies are perfectly elastic.

34. If A impinge on B at rest, and B on C at rest; the velocity communicated to C will be a maximum, when B is a mean proportional between A and C.

For the velocity communicated to $B = \frac{(1+\lambda)Aa}{A+B}$ (Art. 28),

and that by B to $C = \frac{(1+\lambda)B}{B+C} \cdot \frac{(1+\lambda)Aa}{A+B}$, which will be a maximum, when $\frac{(A+B)(B+C)}{B}$ or $B + \frac{AC}{B}$ is a minimum;

$$\therefore\ B^2 = AC.\ \textit{Diff. Calc.}$$

35. Cor. 1. If there be any number of bodies B, C, D, . . . L at rest, and A impinge on B, B on C, C on D, ... the velocity thus communicated to the last will be a maximum, when they are all in geometrical progression.

36. To find the number of mean proportionals interposed between A and L, that the velocity communicated to L may be a maximum.

Let x be the common ratio, $\therefore$ Ax, Ax^2, Ax^3 . . . . . Ax^n are the interposed bodies. $\therefore$ $L = Ax^{n+1}$, and $\therefore$ $n + 1 \propto \frac{1}{\log_e x}$.

But (Art. 28) the velocity communicated to $B = \frac{1+\lambda}{1+x} \cdot a$; $= \frac{\overline{1+\lambda}\cdot A}{A+}$

and $\therefore$ to $C = \left(\frac{1+\lambda}{1+x}\right)^2 \cdot a$ . . . . . . $= \frac{\overline{1+\lambda}^2 \cdot Aa}{(1+x)\cdot(A\lambda + Ax)}$. &c = - - -

and to $L = \left(\frac{1+\lambda}{1+x}\right)^{n+1} \cdot a$, which is to be a maximum.

$$\therefore\ (n+1).\{\log_e(1+\lambda)-\log_e(1+x)\},$$

or, $\dfrac{\log_e(1+\lambda)}{\log_e x}-\dfrac{\log_e(1+x)}{\log_e x}$ is to be a maximum,

$$\therefore\ x^x=\left(\frac{1+x}{1+\lambda}\right)^{1+x}, \text{ by Diff. Calc.}$$

$1+x\cdot L_e\frac{1+\lambda}{1+x} = x\ L_e x$

37. When there are an infinite number of bodies interposed between A and L, to find the velocity communicated to L.

Let $B = A + z$, where z is indefinitely small; then

$$\frac{\text{B's vel.}}{\text{A's vel.}}=\frac{(1+\lambda)\,A}{A+B}=\frac{1+\lambda}{2}\cdot\frac{A}{A+\frac{z}{2}}=\frac{1+\lambda}{2}\cdot\frac{\sqrt{A}}{\sqrt{A+z}}$$

$$=\frac{1+\lambda}{2}\cdot\sqrt{\frac{A}{B}},\text{ nearly.}$$

Similarly,

$$\frac{\text{C's vel.}}{\text{B's vel.}}=\frac{1+\lambda}{2}\cdot\sqrt{\frac{B}{C}};\ \ldots\ldots\ \frac{\text{L's vel.}}{\text{K's vel.}}=\frac{1+\lambda}{2}\cdot\sqrt{\frac{K}{L}}$$

and multiplying all these equations together,

$\dfrac{\text{L's vel.}}{\text{A's vel.}}=\left(\dfrac{1+\lambda}{2}\right)^n\cdot\sqrt{\dfrac{A}{L}}$, which $\because n=\infty$, is $=0$ in every case, except when the bodies are perfectly elastic, and then A's vel. : L's vel. :: $\sqrt{L}:\sqrt{A}$.

38. An imperfectly elastic body impinges obliquely upon a perfectly hard, smooth, and immoveable plane, to find the direction and velocity of its motion after impact.

Def. The angle which the direction of a body's motion makes with a normal to the surface at the point of incidence, is called before impact the *angle of incidence*, and after impact the *angle of reflection*.

Let PC (Fig. 1) in the direction of a body's motion, represent the velocity before impact on the plane AB, CD the normal, PD || AB; divide CD in E, so that $\frac{CE}{CD}=\lambda$; draw EQ || AB and = PD, join CQ. Then the motion of the body before impact may be resolved into the two PD, DC (Art. 20), of which the latter will be entirely destroyed by the impact, and another velocity in the contrary direction $CE = \lambda . CD = CE$ generated by the force of elasticity, hence CE represents the velocity of the body $\perp$ to the plane after impact, and since there is no force acting upon the body during impact $\parallel$ to the plane, $\therefore$ the motion in that direction will remain unaltered, and will be represented by EQ, $\therefore$ CQ is the velocity and direction of the body's motion after impact, $\therefore$ tan $\angle$ incid. : tan $\angle$ refl. :: tan PCD : tan QCE :: $\frac{PD}{CD} : \frac{QE}{CE}$ :: CE : CD :: $\lambda : 1$.

And, velocity before impact : vel. after :: PC : CQ :: PD. cosec $\angle$ of incidence : QE. cosec $\angle$ of refl. :: sin $\angle$ refl. : sin $\angle$ incidence.

39. Cor. 1. If the ball be inelastic, the $\angle$ of refl. $= 90°$, and $\therefore$ the vel. before impact : vel. after :: 1 : sin PCD :: 1 : cos PCA.

$\therefore$ vel. bef. imp. : vel. lost by impact :: 1 : vers PCA.

40. Cor. 2. If the ball in last Cor. impinge very obliquely upon the plane, so that the $\angle$ PCA be indefinitely small, the velocity lost will be an infinitesimal quantity of the second order; for vers PCA $\propto \sin^2 \frac{PCA}{2} \propto (PCA)^2$.

Hence the velocity lost by a body sliding against a smooth curve surface (which may be considered as made up of an infinite number of planes inclined at indefinitely small $\angle^s$ to each other) is an infinitesimal quantity of the first order, and may $\therefore$ be neglected.

41. To find the direction in which an elastic body must be projected from a given point; so that, after reflection at a given plane surface, it may pass through a given point.

Suppose the body is to be projected from A (Fig. 2) against the plane ED, so as to pass through the point B. Draw BC $\perp$ to the plane, making $\frac{BD}{DC} = \lambda$; join A C; it is the direction required. For let it cut the plane in E, and join BE. Project the body in the direction AE;

$$\therefore \frac{\tan \angle \text{ incid.}}{\tan \angle \text{ EBD}} = \frac{\tan \text{C}}{\tan \text{B}} = \frac{\tan \text{BED}}{\tan \text{CED}} = \frac{\text{BD}}{\text{CD}} = \lambda,$$

∴ (Art. 38) EBD = ∠ of reflection; hence, if the body be projected in the direction AE, it will pass through the given point B.

42. To find the direction in which an elastic body must be projected from a given point; so that, after reflection at any number of given plane surfaces, in a given order, it may pass through a given point.

Let A, B (Fig. 3) be the given points, CK, DH, EF ..... the planes in order. Draw BG ⊥ FE, so that $\frac{\text{BF}}{\text{FG}} = \lambda$; GI ⊥ DH, so that $\frac{\text{GH}}{\text{HI}} = \lambda$; IL ⊥ CK, so that $\frac{\text{IK}}{\text{KL}} = \lambda$; join AL, it is the direction required. For join CI, DG, BE; then from Art. 41 it is manifest, that after describing AC the body will be reflected in direction CI; then describe CD and be reflected in direction DG; and, lastly, in direction EB.

43. The velocities, estimated in a given direction, of any number of bodies, whose motions are uniform, being given; find the velocity of their common centre of gravity in the same direction.

Let A, a be the mass, and velocity estimated in the given direction, of one of the bodies, and suppose the axis of $x \parallel$ to the given direction; x, $\dot{x}$ the distances of A, and the common centre of gravity, from plane yz; ∴ $\dot{x}\,\Sigma \text{A} = \Sigma(\text{A}x)$ by the nature of the centre of gravity; and at the end of the interval t, by the same property, if $\dot{x}'$ be the dist. of the centre of gravity from yz, we have $\dot{x}'\,\Sigma\,\text{A} = \Sigma\,\text{A}(x + at) = \Sigma\,\text{A}x + t\,\Sigma\,\text{A}a$, from which equation, subtracting the former and dividing by Σ A, we have $\dot{x}' - \dot{x} = \frac{\Sigma(\text{A}a)}{\Sigma \text{A}} \cdot t$.

Hence (Art. 9) the centre of gravity moves uniformly with a velocity in the given direction $= \dfrac{\Sigma(\mathrm{A}a)}{\Sigma \mathrm{A}}$.

44. Cor. Hence if the bodies describe straight lines in space with uniform motions, and the vel. of one of them (A) be a, b, c in direction of the co-ordinates, the velocity of the centre of gravity will be

$$\sqrt{\left(\frac{\Sigma \mathrm{A}a}{\Sigma \mathrm{A}}\right)^2 + \left(\frac{\Sigma \mathrm{A}b}{\Sigma \mathrm{A}}\right)^2 + \left(\frac{\Sigma \mathrm{A}c}{\Sigma \mathrm{A}}\right)^2}, \text{ and is } \therefore \text{ uniform.}$$

45. The paths of any number of bodies, moving uniformly in straight lines, being given; to find the path of their common centre of gravity.

Let x, y, z be the co-ordinates of A; $\dot{x}, \dot{y}, \dot{z}$ those of the common centre of gravity. The equations of A's path are, $x = at + \alpha$, $y = bt + \beta$, $z = ct + \gamma$, where a, b, c are as before, and α, β, γ are the co-ordinates of A when $t = 0$.

$$\therefore \dot{x}\, \Sigma \mathrm{A} = \Sigma \mathrm{A}(at + \alpha) = t.\, \Sigma \mathrm{A}a + \Sigma \mathrm{A}\alpha.$$

Similarly,

$\dot{y}\, \Sigma \mathrm{A} = t.\, \Sigma \mathrm{A}b + \Sigma \mathrm{A}\beta$, and $\dot{z}\, \Sigma \mathrm{A} = t\, \Sigma \mathrm{A}c + \Sigma \mathrm{A}\gamma$; and by eliminating t from these equations, we obtain

$$\dot{x}\frac{\Sigma \mathrm{A}}{\Sigma \mathrm{A}a} - \frac{\Sigma \mathrm{A}\alpha}{\Sigma \mathrm{A}a} = \dot{y}\frac{\Sigma \mathrm{A}}{\Sigma \mathrm{A}b} - \frac{\Sigma \mathrm{A}\beta}{\Sigma \mathrm{A}b} = \dot{z}\frac{\Sigma \mathrm{A}}{\Sigma \mathrm{A}c} - \frac{\Sigma \mathrm{A}\gamma}{\Sigma \mathrm{A}c}.$$

which are the equations of the straight line in which the centre of gravity moves.

46. Cor. If any of the bodies happen to impinge against each other in the course of their motions, the motion of the centre of gravity will remain unaltered both in velocity and direction.

For whatever moment one body loses in a given direction by impact the other gains in the same direction, and $\therefore$ the whole moment in that direction remains unaltered. Now by (Art. 44) the *velocity*, and by (45) the *direction* of the motion of the centre of gravity depends only * on the sum of the moments, which being unaltered by impact, the Cor. is true.

47. Two bodies move uniformly in straight lines; to find when they are at a given distance from each other.

Let $x = at + \alpha, y = bt + \beta, z = ct + \gamma$; $x' = a't + \alpha'$, $y = b't + \beta', z = c't + \gamma'$; be the equations of their paths; D the given distance.

$$\therefore D^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

$$= \{(a-a')t + \alpha-\alpha'\}^2 + \{(b-b')t + \beta-\beta'\}^2 + \{(c-c')t + \gamma-\gamma'\}^2$$

from which equation two values of t will be obtained.

48. Cor. 1. To find the time when the distance is a minimum we must put $d_t \, D = 0$,

$$\therefore t = \frac{(a-a')(\alpha'-\alpha) + (b-b')(\beta'-\beta) + (c-c')(\gamma'-\gamma)}{(a-a')^2 + (b-b')^2 + (c-c')^2}.$$

49. Cor. 2. If we have to find the time and place of impact of two spheres, we must make $D =$ the sum of their radii, and take the *least* value of t, for that is the time of their surfaces first touching; and after that their directions and velocities are altered so that the above equation is no longer applicable; and ∴ the second root cannot be used.

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* It may be urged, that it depends also on the quantities $\Sigma(A\alpha)$, $\Sigma(A\beta)$, $\Sigma(A\gamma)$; but these are dependent only on the moments, for $\frac{\Sigma A\alpha}{\Sigma A} =$ distance of the common centre of gravity from plane yz when $t = 0$; and since the *velocity* of the centre of gravity is uniform, and remains unaltered after impact, it is clear that howsoever α may be altered by impact, yet $\frac{\Sigma A\alpha}{\Sigma A}$ and ∴ $\Sigma A\alpha$ remains constant. Similarly, $\Sigma A\beta$, $\Sigma A\gamma$ are constant.

CHAPTER III.

ON THE MOTION OF BODIES WHEN ACTED ON BY UNIFORMLY ACCELERATING OR RETARDING FORCES.

50. If a body be acted on by a uniform accelerating force f; and $v =$ the velocity generated in the time t; then $v = ft$.

For (Art. 13) the velocity generated in each second $= f$, and $\therefore$ velocity generated in time $t = ft$.

51. If $u =$ the velocity when $t = 0$, the velocity [redacted] time $t = u \pm ft$; $\pm$ according as the [redacted] retards the motion.

52. The space desc[redacted], acted on by a uniformly accelerat[redacted] the space it would uniformly describe, in the same time, with the last acquired velocity.

$s =$ the space described, and $v =$ vel. acquired, in the time t, by the action of the uniform accelerating force f. Divide t into (n) intervals, each equal to $\frac{t}{n}$; then vel. acquired in each interval $= \frac{v}{n}$; and the space described in time t with the velocity at the *end* of each interval continued uniform during that interval $= \frac{vt}{n^2} + \frac{2vt}{n^2} + \frac{3vt}{n^2} + \ldots$ to n terms $= \frac{vt}{2}\left(1 + \frac{1}{n}\right)$: $\frac{vt}{n^2}$ and the space described in the time t with the vel. at the *beginning* of each interval continued uniform during that interval $= 0 + \frac{vt}{n^2} + \frac{2vt}{n^2} + \ldots$ to n terms $= \frac{vt}{2}\left(1 - \frac{1}{n}\right)$. Now s manifestly lies between these two spaces, (for it is described with velocities which are always intermediate to the velocities with which they are described) whose difference $\frac{vt}{n}$ can be made less than any assignable quantity by increasing the number of intervals without limit,

$\therefore\ s = \frac{vt}{2} = \frac{1}{2}$ space described uniformly in time t with vel. v. (Art. 8).

53. Cor. 1. By uniting this equation with that in Art. 50, we obtain,

$$v = ft,\ 2s = vt = ft^2,\ v^2 = 2fs,$$

which are the principal equations for determining the motion of a body acted on by a uniform accelerating or retarding force.

54. Cor. 2. The spaces described by the body, from rest, in equal successive portions of time, are as the odd numbers; 1, 3, 5, 7 ....

For the space described in (n) seconds $= \frac{1}{2}fn^2$

$\ldots\ldots\ldots\ (n-1)\ldots = \frac{1}{2}f(n-1)^2.$

$\therefore$ the difference of these, or the space described in the $(n)^{th}$ second $= \frac{1}{2}f(2n-1)$.

$\therefore$ space described in 1^{st} second $= \frac{1}{2}.f.1$, in $2^d = \frac{1}{2}f.3$, in $3^d = \frac{1}{2}f.5$, ... which are as the Nos. 1, 3, 5, ...

55. Cor. 3. If the body, instead of beginning to move from rest, be projected in the direction of the force with a vel. u, the space described in time $t = ut \pm \frac{1}{2}ft^2$, the top or bottom sign being used according as the force accelerates or retards the motion.

For $\frac{1}{2}ft^2$ is the space which a body would describe in the time t, by the action of the force f; and ut is the space that would be described by the body with the velocity of projection; and, by Art. 17, the former will not be affected by the latter; $\therefore\ ut \pm \frac{1}{2}ft^2 =$ the space described from both causes.

56. Cor. 4. If a body describe the straight line ABC from rest at A, by the action of a uniform force; we have

$$\begin{array}{rl} (\text{vel.})^2 \text{ at C} &= 2f.\,\text{AC} \\ \underline{(\text{vel.})^2 \text{ at B}} &\underline{= 2f.\,\text{AB}} \\ \therefore\ (\text{vel.})^2 \text{ at C} - (\text{vel.})^2 \text{ at B} &= 2f.\,\text{BC} \\ &= (\text{vel.})^2 \text{ thro' BC from rest at B.} \end{array}$$

57. The force of gravity is a uniform accelerating force, which acts in a direction perpendicular to the horizon.

For it appears by experiments made on heavy bodies, descending freely, that the motion is always $\perp$ to the horizon; and that in every second of time, an additional velocity of nearly 32.18 feet is generated by the force of gravity.

The accelerating force of gravity is usually denoted by the letter g.

58. The force accelerating a body down an inclined plane is a uniform force $= g$. sin plane's inclination to the horizon.

For let AB (Fig. 4) be the plane, BC $\parallel$ the horizon, AC $\perp$ to it. Draw CD $\perp$ AB; then AC being taken to represent the force of gravity, AD represents that part which is effective upon the body, in accelerating it down the plane, DC being destroyed by the reaction of the plane,

$\therefore$ force down the plane $= \text{AD} = \text{AC}.\ \sin \text{ACD} = g.\ \sin \text{B}$.

Hence all the formulæ which have been deduced for constant forces are applicable to motion in vertical lines, and upon inclined planes, when the body is acted on by gravity.

59. The velocity acquired from rest down an inclined plane, is equal to that acquired down its altitude.

$$\begin{array}{rl} \text{For } (\text{vel.})^2.\ \text{acquired down AB} &= 2 \text{ force} \times \text{space} \\ &= 2g.\ \sin \text{B}.\ \text{AB} \\ &= 2g.\ \text{AC} \\ &= (\text{vel.})^2 \text{ acquired down AC.} \end{array}$$

60. If chords be drawn from the extremity of a vertical diameter of a circle, the velocities acquired down them from rest will vary as their lengths, and the times of descent will be equal.

Let AB (Fig. 5) be a vertical diameter, CB a chord; CD $\perp$ AB; then vel. down CB $= \sqrt{2g.\overline{\mathrm{BD}}}$ (last Prop.) $\propto \sqrt{\overline{\mathrm{BD}}} \propto \sqrt{\overline{\mathrm{AB}.\,\mathrm{BD}}} \propto$ CB. and the time down CB $\propto \frac{\text{space}}{\text{vel.}}$ (Art. 52) $\propto \frac{\mathrm{BC}}{\mathrm{BC}} \propto 1$, and is $\therefore$ invariable for all chords in the circle, drawn from the highest or lowest point.

61. The property of the circle just demonstrated is very useful in solving problems concerning planes of quickest and slowest descent, as the following example will shew:

To find the plane of quickest descent from a point within a vertical circle to its circumference.

Let D (Fig. 6) be the given point, C the centre of the circle, BCG a vertical diameter. Join GD and produce it to meet the circumference in E, DE is the plane required. For join CE, and draw DF $\parallel$ BG. Then $\because$ EC $=$ CG $\therefore$ EF $=$ FD, $\therefore$ a circle described with centre F and radius FE will pass through E, D, and touch the given circle in E, $\because$ CE passes through their centres: hence any plane from D to the circumference of the given circle, except DE, will fall partly without the circle DE*d*, and $\therefore$ the time down DE will be less than down any other.

62. If a body descend from rest, by the action of gravity, down a curve line, situated in a vertical plane; the velocity acquired will be the same as if it had fallen in a straight line through the same vertical space.

For let ABCD . . . (Fig. 7) be a system of smooth planes, inclined to each other at indefinitely small $\angle^s$, so that (Art. 40) no vel. may be lost in passing from one to another; draw A*a*, B*b*, C*c*, ... horizontal, and *a b c d* . . . . vertical; produce CB, DC . . . to F, G . . . . . Then the velocity acquired down AB = that down *ab* (Art. 59) = that down FB; ∴ the body begins to descend down BC as if it had fallen at once down FB, ∴ vel. down AC = that down FC = that down *ac* = that down GC; and, similarly, vel. down AD = that down GD = that down *ad*; . . . . . . Hence the vel. down the system of planes = that down the altitude; let now the number of planes be increased, and the length of each diminished without limit, and the system of planes becomes a curve line, in which the same is true.

63. Cor. 1. Hence a body acquires the same velocity in descending from one given horizontal plane to another, whatever be the form of its path.

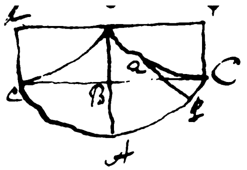
64. Cor. 2. If a body be projected up a curve, the perpendicular height to which it will rise is equal to that through which it must fall to acquire the vel. of projection.

65. Cor. 3. If B (Fig. 8) be the lowest point of the curve ABC, of which the branches BA, BC are similar and equal, a body in falling down AB will ascend to C; and since the velocities are equal at all equal altitudes in the ascent and descent, the whole time of ascent will be equal to the whole time of descent.

66. Cor. 4. The same things are true, if the body instead of sliding down the curve, be made to describe the curve line, by means of a string, which is at every point ⊥ to it. For the string will sustain that part of the weight which was before sustained by the curve. In this case the body is said to *oscillate.*

67. To make a body oscillate in a given cycloid.

Let APC*c* (Fig. 9) be a given cycloid, AB its axis, C*c* its horizontal base. Produce AB to S making BS = AB;



complete the parallelogram CDS*dc*, and upon SD, S*d* as bases, and CD, *cd* as axes, describe two semicycloids SC, S*c*. By *Diff. Calc.*, CS*c* is the evolute of CA*c*, and if PE be a tangent at E, PE = EC, and is ⊥ to curve at P. Hence if the string SEC, fixed at S, and wrapped along the semicycloid SEC, be unwrapped, beginning at C, its extremity will describe the semicycloid CPA, and in wrapping itself on the semicycloid S*c*, its extremity will complete the cycloid CA*c*; hence a body P suspended by the string SEP will oscillate in the cycloid CA*c*.

68. To find the time of oscillation in a cycloid. vide page 57

Let ACB (Fig. 10) be the cycloid, as before; and suppose the body to begin to descend from L; let MN be a small arc described; draw LR, MT, NU horizontal, and on AR describe the semicircle A*m*R, join A*m*, A*n*, R*m*, make A*o* = A*n*. Then

$$\text{vel. at M} = \sqrt{2g.\,\text{RT}} = \sqrt{\frac{2g}{\text{AR}}}.\ \text{R}m$$

$$\text{and arc MN} = 2\,(\sqrt{\text{AB. AT}} - \sqrt{\text{AB. AU}}),\ \text{by } \textit{Diff. Calc.}$$

$$= 2\sqrt{\frac{\text{AB}}{\text{AR}}}.\,(\sqrt{\text{AR. AT}} - \sqrt{\text{AR. AU}})$$

$$= 2\sqrt{\frac{\text{AB}}{\text{AR}}}.\ (\text{A}m - \text{A}n) = 2\sqrt{\frac{\text{AB}}{\text{AR}}}.\ mo$$

$$\therefore \text{time through MN} = \frac{\text{MN}}{\text{vel. at M}}\ \text{ultimately,}$$

$$= \sqrt{\frac{2.\,\text{AB}}{g}}.\,\frac{mo}{\text{R}m} = \sqrt{\frac{2.\,\text{AB}}{g}}.\,\frac{mn}{\text{AR}}\ \text{ultimately, by similar } \triangle^{s}.$$

$$\therefore \text{whole time from L to A} = \sqrt{\frac{\text{AS}}{g}}.\,\frac{\text{R}mn\text{A}}{\text{AR}} = \frac{\pi}{2}\sqrt{\frac{\text{AS}}{g}}.$$

$$\therefore \text{time of an oscillation} = \pi\sqrt{\frac{\text{AS}}{g}}.$$

69. We have now considered the motion of a body, acted on by gravity, in straight lines; and when constrained to move

along a certain curve; we have next to consider the path it would take if left to itself.

A body is projected in vacuo, from a given point, in a given direction, with a given velocity, required its path.

Let it be projected from A (Fig. 11) in direction AC, and describe the curve APB. P its place at time t. Let Ax horizontal, and Ay vertical, be the co-ordinate axes; draw CPD ∥ Ay. $x =$ AD, $y =$ PD, $v =$ vel. projection, CAD $= \theta$. Then velocity in the direction AD will be constant, and $= v \cos\theta$, ∵ there is no force upon the body in that direction,

$$\therefore x = vt \cos\theta;$$

and (Art. 11) the body would have been at C at time t, ∴ CP is the space due to force of gravity (Art. 17), ∴ CP $= \frac{1}{2} . gt^2$. But CD $=$ AD $\tan\theta = vt . \sin\theta$.

$$\therefore y = vt \sin\theta - \tfrac{1}{2} gt^2,$$

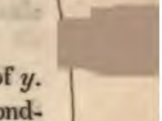
∴ by eliminating t, we have

$$y = x \tan\theta - \frac{gx^2}{2v^2} \cdot \sec^2\theta,$$

which is the equation of the path, and shews that it is a parabola.

70. **Cor. 1.** If $h =$ altitude due to the velocity of projection, $\frac{2v^2}{g} = 4h$;

$$\therefore y = x \tan\theta - \frac{x^2}{4h} . \sec^2\theta.$$



71. **Cor. 2.** By transposition, division, and completing the square, we have

$$(x - h \sin 2\theta)^2 = 4h \cos^2\theta \, (h \sin^2\theta - y).$$

$$\therefore \text{Lat. rectum} = 4h \cos^2\theta,$$

and greatest altitude $= h \sin^2\theta$, being the greatest value of y. and the range AB $= 2h \sin 2\theta$, being twice the correspond- in[illegible] x.

Sin[illegible]st altitude + $\frac{1}{4}$ Latus rectum = dit. di [illegible] ∵. the velocity at any point is [illegible]e directrix of the parabola.

73. Cor. 4. Since range $= 2h \sin 2\theta$, if the vel. of projection be given, the range will be a maximum when $\theta = 45°$.

74. To find the length of the curve described.

$$\because y = x \tan\theta - \frac{x^2}{4h} \cdot \sec^2\theta$$

$$\therefore p = d_x y = \tan\theta - \frac{x}{2h} \cdot \sec^2\theta, \; \therefore d_p x = -2h\cos^2\theta,$$

$$\therefore s = \textstyle\int_x \sqrt{1+p^2} = \int_p \sqrt{1+p^2}.\, d_p x = -2h\cos^2\theta . \int_p \sqrt{1+p^2}$$

$$= -h\cos^2\theta \,\{C + p\sqrt{1+p^2} + \log_e (p + \sqrt{1+p^2})\}$$

or, if $\tan\phi = p =$ tan of inclination of the curve to the horizon,

$$= h\cos^2\theta \,\{\tan\theta . \sec\theta + \log_e(\sec\theta + \tan\theta) - \tan\phi . \sec\phi + \log_e(\sec\phi - \tan\phi)\}$$

$$= h\cos^2\theta \,\{\tan\theta . \sec\theta + \log_e \tan(45° + \frac{\theta}{2}) - \tan\phi . \sec\phi + \log_e \tan(45° - \frac{\phi}{2})\}$$

$$\therefore \text{the curve APB} = 2h\cos^2\theta . \{\tan\theta . \sec\theta + \log_e \tan(45° + \frac{\theta}{2})\}$$

$$= 2h\,\{\sin\theta + \cos^2\theta \log_e \tan(45° + \frac{\theta}{2})\}.$$

75. Cor. 1. If the velocity be given, and the direction of projection be required, that the length of the curve APB may be a maximum, we must put the differential coefficient of the last expression $= 0$.

$$\therefore \operatorname{cosec}\theta = \log_e \tan\left(45° + \frac{\theta}{2}\right) \; *$$

from which equation θ must be found by approximation.

The length of the curve APB $= 2h \operatorname{cosec}\theta$, is then known also.

76. The following problems are intended to illustrate the foregoing principles.

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* This expression may be put under any one of the forms

$$\sec\theta = \frac{e^{\operatorname{cosec}\theta} + e^{-\operatorname{cosec}\theta}}{2}; \; \tan\theta = \frac{e^{\operatorname{cosec}\theta} - e^{-\operatorname{cosec}\theta}}{2}; \; \operatorname{cosec}\theta = \frac{e^{\operatorname{cosec}\theta} + e^{-\operatorname{cosec}\theta}}{e^{\operatorname{cosec}\theta} + e^{-\operatorname{cosec}\theta}}.$$

Ex. 1. Required in what direction a body must be projected from a given point in an inclined plane, that the range may be a maximum.

Let α = the inclination of the plane to the horizon, x, y co-ordinates of the point where the body strikes the plane.

$$\therefore\ y = x \tan \alpha,$$

$$\therefore\ x \tan \alpha = x \tan \theta - \frac{x^2}{4h}(1 + \tan^2 \theta);$$

now x is to be a maximum, $\therefore\ d_\theta x = 0$; hence, dividing the above equation by x, and differentiating with regard to $\tan \theta$, considering x constant, we have

$$0 = 1 - \frac{x}{2h}.\tan \theta,$$

$$\therefore\ \tan \alpha = -\cot 2\theta, \text{ by eliminating } x;$$

$$\text{and } \therefore\ \cos(2\theta - \alpha) = 0 = \cos\frac{\pi}{2},$$

$$\therefore\ \theta - \alpha = \frac{90^\circ - \alpha}{2};$$

which shews that the direction of projection bisects the ∠ between the given plane and the vertical.

Ex. 2. From the top of a tower two bodies are projected with the same given velocity, at different angles of elevation, and strike the horizon at the same point; to find the height of the tower.

Take the top of the tower for the origin of co-ordinates, y = its altitude, and suppose the ball strikes the plane at the distance x from the bottom; α, β the ∠s of elevation of the directions of projection, h = altitude due to the vel. of projection.

$$\therefore\ -y = x \tan \theta - \frac{x^2}{4h}(1 + \tan^2 \theta),$$

$$\therefore\ \tan^2 \theta - \frac{4h}{x}\tan \theta - \frac{4hy}{x^2} + 1 = 0;$$

and the roots of this equation are $\tan \alpha$, $\tan \beta$; hence, by the nature of equations,

$$\tan\alpha + \tan\beta = \frac{4h}{x},$$

$$\text{and} \tan\alpha\tan\beta = 1 - \frac{4hy}{x^2} = 1 - \frac{y}{4h}(\tan\alpha + \tan\beta)^2,$$

$$\therefore y = 4h.\cot(\alpha+\beta).\frac{\cos\alpha\cos\beta}{\sin(\alpha+\beta)}.$$

Ex. 3. An elastic body falls from a given altitude upon a perfectly hard horizontal plane, and rebounds and descends again, and so on till the motion ceases; required the whole space described.

a = the given altitude, λ = the elasticity of the body.

$$\therefore \text{vel. acquired in } 1^{st} \text{ descent} = \sqrt{2g.a.}$$

$$\therefore \text{vel. of ascent} = \lambda\sqrt{2ga.}$$

$$\therefore \text{space described in } 1^{st} \text{ ascent} = \frac{2ga\lambda^2}{2g} = a\lambda^2.$$

$$\text{Similarly, in } 2^{d} \text{ ascent} = a\lambda^4,$$

$$\ldots\ldots\ldots\ldots = \ldots$$

$\therefore$ whole space described

$$= a(1 + 2\lambda^2 + 2\lambda^4 + 2\lambda^6 + \ldots\ldots \textit{ad infin.})$$

$$= a.\frac{1+\lambda^2}{1-\lambda^2}.$$

Ex. 4. A body is projected from a given point, with a given velocity; to find the direction that the area of the parabola described may be a maximum.

The range $= 2h.\sin 2\theta$, and the greatest altitude $= h\sin^2\theta$, $\therefore$ area $\propto \sin 2\theta.\sin^2\theta \propto \sin 2\theta.(1-\cos 2\theta) \propto 2\sin 2\theta - \sin 4\theta$, a maximum, which will be the case when

$$\cos 4\theta = \cos 2\theta = \cos(360^\circ - 2\theta),$$

$$\therefore 4\theta = 360^\circ - 2\theta, \therefore \theta = 60^\circ.$$

Ex. 5. A body falling from the top of a tower was observed to fall $\left(\frac{1}{n}\right)^{th}$ of its altitude in the last second; to find the whole time of descent and altitude of the tower.

Let t = time of descent, $\therefore$ altitude of the tower $= \frac{1}{2}gt^2$,

and space descended in the last second $= \frac{1}{2}g\,(2t-1)$;

$$\therefore\ \tfrac{1}{2}gt^2 = \tfrac{1}{2}.\,g.\,n\,(2t-1),$$

$$\therefore\ t = n - \sqrt{n^2 - n},$$

$$\therefore\ \text{altitude} = \tfrac{1}{2}.\,gn\,(2n - 1 - 2\sqrt{n^2 - n}).$$

Ex. 6. Two bodies are projected from the same point with equal velocities, so that their ranges on a horizontal plane are equal; to find the angles of projection, that the area included between the parabolas described may be a maximum.

$$\text{The range} = 2h \sin 2\theta = 2h \sin (180^\circ - 2\theta),$$

$\therefore$ the $\angle^s$ of projection are θ and $90^\circ - \theta$.

Now the area of the parabola described by the first $\propto \sin 2\theta.\ \sin^2\theta$, and by the second $\propto \sin (180^\circ - 2\theta) \sin^2 (90^\circ - \theta)$ $\propto \sin 2\theta \cos^2\theta$,

$\therefore$ the area included $\propto \sin 2\theta.\ (\cos^2\theta - \sin^2\theta) \propto \sin 4\theta$, a maximum.

$$\therefore\ \sin 4\theta = 1 = \sin 90^\circ.$$

$$\left.\begin{array}{r} \therefore\ \theta = 22^\circ\tfrac{1}{2}. \\ \text{and } 90^\circ - \theta = 67^\circ\tfrac{1}{2}. \end{array}\right\} \text{the } \angle^s \text{ of projection.}$$

Ex. 7. An elastic body slides down an inclined plane of given length, and impinges on the perfectly hard horizontal plane, on which the base of the inclined plane rests; to determine the inclination, that the range of the body on the horizontal plane after reflection may be a maximum.

a = the length of the plane, θ = its inclination, θ' = the $\angle$ at which the body is reflected after impact, λ = its elasticity. $\therefore \tan \theta' = \lambda \tan \theta$; and the altitude of the plane $= a \sin \theta$; $\therefore$ the body strikes the plane with a velocity $= \sqrt{2ga \sin \theta}$,

and is reflected with a velocity $= \dfrac{\cos\theta}{\cos\theta'}.\sqrt{2ga\sin\theta}$, (Art. 38).

and the altitude due to this velocity $= a.\dfrac{\cos^2\theta.\ \sin\theta}{\cos^2\theta'}$.

$$\therefore \text{ the range} = 2a.\,\frac{\cos^2\theta.\,\sin\theta}{\cos^2\theta'}.\,\sin 2\theta',\text{(Art. 71)}.$$

$$= 4a\lambda.\,(\cos\theta - \cos^3\theta).\ \text{a maximum,}$$

$$\therefore\ \theta = \sec^{-1}.\,\sqrt{3},\ \text{and the range} = \frac{8a\lambda}{3\sqrt{3}}.$$

Ex. 8. **An infinite number of bodies are projected from a given point, with the same velocity, in all directions; to find the locus of them all at a given time.**

Let AB, AB_1, AB_2, ... (Fig. 12) be the spaces that would be described in any time with the velocity and directions of projection of the bodies P, P_1, P_2, . . . ; these spaces are all equal, ∵ the velocity of projection is the same for all; draw PB, P_1B_1, P_2B_2, ... vertical; these also are all equal, ∵ they are the spaces due to the action of gravity on the bodies during the time from projection. Complete the parallelograms BO, B_1O, B_2O, ... ∴ OP $=$ OP_1 $=$ $OP_2 =$ . . . . hence the bodies are situated in the surface of a sphere, whose centre is O.

Ex. 9. In the last example, to find the locus of the vertices of all the parabolas described.

Let x, y be co-ordinates of the vertex of any one of them.

$$\therefore\ x = h\sin 2\theta,\ y = h\sin^2\theta.$$

$$\therefore\ x^2 = 4h^2\sin^2\theta.\cos^2\theta = 4y(h-y) = 4(hy-y^2);$$

which is the equation of an ellipse, whose axes are $2h$, and h. ∴ the surface required will be an oblate spheroid.

Ex. 10. To find the surface which envelopes all the parabolas described.

$$y = x\tan\theta - \frac{x^2}{4h}(1 + \tan^2\theta);$$

and differentiating this expression with regard to tan θ, (which is the parameter), considering x, y as constant, we have

$$0 = x - \frac{x^2}{2h}\tan\theta, \quad \therefore \tan\theta = \frac{2h}{x},$$

$$\therefore y = 2h - \frac{x^2}{4h} - h,$$

$$\therefore x^2 = 4h(h - y).$$

which is the equation of a parabola; and ∴ the surface required will be a paraboloid of revolution about a vertical axis passing through the point of projection.

CHAPTER IV.

ON THE FREE MOTION OF A PARTICLE OF MATTER WHEN ACTED ON BY VARIABLE FORCES.

77. If s be the space described by a particle of matter, v its velocity, and f the accelerating force acting on it at the time t, reckoning from some fixed epoch; then $v = d_t s$, and $f = d_t v = d_t^2 s$.

For let $v + \delta v$, $s + \delta s$, $f + \delta f$ be the velocity, space, and accelerating force at the time $t + \delta t$; $\therefore$ the space δs is described in the time δt, with velocities varying continually from v to $v + \delta v$,

$$\text{and } \therefore \delta s \text{ lies between } v\delta t \text{ and } (v + \delta v)\,\delta t,$$

$$\therefore \frac{\delta s}{\delta t} \;.\;.\;.\;.\; v \;.\;.\; v + \delta v,$$

and taking the limits, $d_t s = v$.

Again, the velocity δv is acquired in the time δt, by the action of accelerating forces varying continually from f to $f + \delta f$;

$$\text{and } \therefore \delta v \text{ lies between } f\delta t \text{ and } (f + \delta f)\,\delta t,$$

$$\therefore \frac{\delta v}{\delta t} \;.\;.\;.\; f \;.\;.\; f + \delta f,$$

and taking the limits as before, $d_t v = f$.

By eliminating v from this equation $f = d_t^2 s$.

78. Cor. $f = d_t v = d_t s \,.\, d_s v = v d_s v$. It is the integral of this equation which is geometrically constructed by Newton, in his 7th sect. prop. 39.

79. In Chap. III. it was shewn that gravity is a uniform force, but this is only true for small altitudes above the Earth's surface; it will be shewn hereafter, in the *Treatise on Attractions*, that the accelerating force of a sphere on a particle situated *without* it, varies inversely as the square of its distance from the centre; and on a particle *within* it, directly as its distance from the centre.

Ex. 1. Let it be required to find the relations between the space, time, and velocity of a particle, falling from a considerable altitude towards the Earth's centre, neglecting the resistance of the air.

$2a$ = distance of the point, from which the body begins to fall, from the Earth's centre, s = the distance of the particle from the centre, and v = velocity acquired at time t; μ = the Earth's absolute force; or the attraction its mass, collected at its centre, would exert on a particle at distance 1.

$$\therefore\ vd_tv = \frac{-\mu}{s^2},$$

$$\therefore\ v^2 = 2\mu\left(\frac{1}{s} - \frac{1}{2a}\right) = \mu.\frac{2as - s^2}{as^2}\ \ldots\ldots (1).$$

$$\therefore\ t = \sqrt{\frac{a}{\mu}}\int \frac{-s}{\sqrt{2as - s^2}}\ \ldots\ (\text{Art. } 77). \qquad = \int \frac{a - s + a}{\sqrt{2as - s^2}} ds$$

$$= \sqrt{\frac{a}{\mu}}.\left\{\sqrt{2as - s^2} + a\ \text{vers}^{-1}\frac{2a - s}{a}\right\}\ \ldots\ (2). \qquad \int ds \frac{a - s}{\sqrt{2as - s^2}}$$

Equations (1) (2) determine the relations required. $\sqrt{2as - s^2}$

Cor. 1. If we suppose the Earth's mass collected at its centre, we have the whole time of the particle's falling to the

$$\text{centre} = \frac{a^{\frac{3}{2}}\pi}{\sqrt{\mu}}. \qquad \sqrt{\frac{a}{\mu}}\int_{s=0,\ 2a} \frac{-s}{\sqrt{2as - s^2}} \quad \left(\tfrac{1}{2}a\pi\right)$$

Cor. 2. We may construct for the velocity and time, thus—

Let AC (Fig. 13) be the whole space to the centre; on it describe the semicircle ADC, and draw the tangent AT, and PD ⊥ AC, join CD.

Then the velocity acquired through AP

$$\propto \frac{\sqrt{2as - s^2}}{s} \propto \frac{\text{PD}}{\text{CP}} \propto \text{AT},$$

$\text{PD}^2 = \overline{2a - s}\cdot s = 2as - s$

$\frac{\text{PD}^2}{\text{CP}^2} = \frac{2as - s^2}{s^2}$

$\therefore \frac{\text{PD}}{\text{CP}} \propto \frac{\sqrt{2as - s}}{s}$

and the time of falling through AP

$$\propto \sqrt{2as - s^2} + a \text{ vers}^{-1}\frac{2a - s}{a},$$

$$\propto \text{PD} + \text{arc AD}, \quad \text{vers}^{-1}\frac{s}{a}$$

$$\propto \tfrac{1}{2}\,\text{AC}.\ (\text{PD} + \text{arc AD}),$$

$$\propto \text{area ACD}.$$

This is the construction given by Newton, sect. vii. prop. 36.

Ex. 2. Let it be required to find the relations between the space, time, and velocity of a particle falling from a point within the Earth directly towards its centre.

a = the distance of the point, from which the particle falls, from the Earth's centre, s = the distance of the particle from the centre at the time t, v = its velocity, μ = the force exerted on a particle at the distance 1.

$$\therefore\ v d_s v = -\mu s,$$

$$\therefore\ v^2 = \mu\,(a^2 - s^2)\ \ldots\ldots\ (1).$$

$$\therefore\ t = \frac{1}{\sqrt{\mu}}\cdot\int_s \frac{-1}{\sqrt{a^2 - s^2}} = \frac{1}{\sqrt{\mu}}\cdot\cos^{-1}\frac{s}{a}\ldots(2).$$

The whole time to the centre $= \frac{\pi}{2\sqrt{\mu}}$, which is independent of the altitude from which the body falls.

We may construct for the velocity and time, thus—

Let S the centre of the semicircle ADC (Fig. 13) be the centre of force, A the point from which the body begins to fall.

$$\text{The velocity at P} \propto \sqrt{a^2 - s^2} \propto \text{PD},$$

$$\text{and time through AP} \propto \cos^{-1}\frac{s}{a} \propto \text{arc AD}.$$

This construction is given by Newton in his 7th sect. prop. 38.

0 Ex. 3. To find the whole time of descent to the centre, when the force varies inversely as the distance.

Using the same notation as in the last example, we have

$$vd_sv = -\frac{\mu}{s},$$

$$\therefore v^2 = 2\mu \log_e\left(\frac{a}{s}\right) = 2\mu\, x^2, \text{ supposing } x^2 = \log_e\left(\frac{a}{s}\right);$$

$$\therefore s = ae^{-x^2}, \text{ and } d_xt = -\frac{d_xs}{v} \doteq a\sqrt{\frac{2}{\mu}}\cdot e^{-x^2},$$

$$\therefore t = a\sqrt{\frac{2}{\mu}}\cdot\int_x^{\prime} e^{-x^2},$$

the integral being taken from $s = a$ to $s = 0$, or from $x = 0$ to $x = \infty$.

Now

$$\left(1-\frac{x^2}{n}\right)^n = 1-x^2+\frac{x^4}{1.2}\cdot\left(1-\frac{1}{n}\right)-\frac{x^6}{1.2.3}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\&c.$$

$$= e^{-x^2}+\frac{1}{n}\cdot\phi x, \text{ where } \frac{1}{n}\cdot\phi x \text{ vanishes when } n \text{ is } \infty,$$

whatever be the value of x.

Let $x^2 = n\cos^2\theta$

$$\therefore \int_{x=0,\infty}^{\prime}\left\{e^{-x^2}+\frac{1}{n}\cdot\phi x\right\} = \int_{x=0,\infty}^{\prime}\left(1-\frac{x^2}{n}\right)^n$$

$$= \sqrt{n}\cdot\int_{\theta=0,\frac{\pi}{2}}\sin^{2n+1}\theta,$$

$$= \sqrt{n}\cdot\frac{2.4.6\ldots\ldots2n}{3.5.7\ldots(2n+1)}$$

$$= \sqrt{\frac{n}{2n+1}}\cdot\sqrt{\left(\frac{2^2}{1.3}\cdot\frac{4^2}{3.5}\cdot\frac{6^2}{5.7}\cdots\frac{(2n)^2}{(2n-1)(2n+1)}\right)}$$

and $\therefore$ because n is infinite

$$\int_{x=0,\infty}^{\prime} e^{-x^2} \doteq \sqrt{\frac{n}{2n+1}\cdot\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}; \text{ by Wallis's theorem.}$$

$$\therefore \text{ the time to the centre } = a\sqrt{\frac{\pi}{2\mu}}.$$

80. To determine the motion of a free particle (m) when acted on by any forces whatever.

Let all the forces be resolved in directions ǁ to the rectangular co-ordinate axes of x, y, z; let X, Y, Z denote the sum of the resolved parts respectively $\parallel x, y, z$. Then, instead of the original forces, we may substitute the forces X Y Z; and, by Art. 17, each of these will produce the same effect as if the other two did not act, and

$$\therefore\ d_t^2x = \frac{X}{m},\ d_t^2y = \frac{Y}{m},\ d_t^2z = \frac{Z}{m},\ \text{by (Arts. 16, 77)} \ldots \text{(A)},$$

and $\because$ the axes are rectangular, $\therefore$ d_tx, d_ty, d_tz (Art. 77) are the velocities of the particle in directions of the co-ordinates.

81. Equations (A) being integrated, give $x = \mathrm{F}t$, $y = ft$ $z = \phi t$; from which, eliminating t, we obtain $\mathrm{F}^{-1}x = f^{-1}y = \phi^{-1}z$ the equations of the path of the particle; which is $\therefore$ in general a curve of double curvature.

82. Multiplying (A) respectively by $2dx$, $2dy$, $2dz$, adding and integrating, we find

$$m\,(d_tx^2 + d_ty^2 + d_tz^2) = \mathrm{C} + 2\int(\mathrm{X}\,dx + \mathrm{Y}dy + \mathrm{Z}dz).$$

or, by Art. 20, $mv^2 = \mathrm{C} + 2\int(\mathrm{X}\,dx + \mathrm{Y}dy + \mathrm{Z}dz)$.

Suppose the quantity $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ to be a complete differential of a function of three variables, $mv^2 = \mathrm{C} + 2\phi(x, y, z)$; or if V be the velocity of the particle at the point a, b, c; $mv^2 - m\mathrm{V}^2 = 2\phi\,(x, y, z) - 2\phi\,(a, b, c)$.

From which we conclude, that the *vis viva* acquired in passing from one point to another, depends only on the co-ordinates of those points, and not at all upon the path taken between them. (Newton, sect. viii. prop. 40.)

When $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ is a complete differential of a function of three variables, we have $d_y\mathrm{X} = d_x\mathrm{Y}$, $d_z\mathrm{Y} = d_y\mathrm{Z}$, $d_x\mathrm{Z} = d_z\mathrm{X}$, and $\therefore$ X Y Z are functions of the co-ordinates only, and do not depend on the time.

83. If the particle be acted on by any number of attractive forces directed towards *fixed* centres, $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ will be a complete differential.

For let a, b, c be the co-ordinates of one of the centres of force (F): r = distance of m from it,

$$\therefore\ r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2,$$

$$\therefore\ d_xr = \frac{x-a}{r},\ d_yr = \frac{y-b}{r},\ d_zr = \frac{z-c}{r}.$$

Hence, for this force,

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = \mathrm{F}\,(d_xr\,.\,dx + d_yr.dy + d_zr\,.\,dz) = \mathrm{F}dr,$$

and $\therefore$ for any number of forces directed to fixed centres,

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = \Sigma(\mathrm{F}dr).$$

But F being a function of r, the right hand member, and $\therefore$ $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$, is a complete differential of a function of r, r', r'' .... &c., that is, of $x, y, z;\ x', y', z';$ ....

O 84. The path taken by the particle will be such, that $\int(mvds)$, or $\int_t(mv^2)$ is less than if it had taken any other path between the same extreme points.

$$\text{For } v^2 = d_ts^2 = d_tx^2 + d_ty^2 + d_tz^2,$$

and taking the variation, $v\delta v = d_t^2x\,.\,\delta x + d_t^2y\,.\,\delta y + d_t^2z\,.\,\delta z.$

But $\because$ t is a function of the three variables $x\ y\ z$, $\therefore$ d_t^2x denotes the *complete* second differential coefficient of x, and is therefore the same as d^2x, divided by dt^2,

$$\therefore\ ds\delta v = vdt\,.\,\delta v = dd_tx\,.\,\delta x + dd_ty\,.\,\delta y + dd_tz\,.\,\delta z\,..\,(1).$$

But, from the equation,

$$ds^2 = dx^2 + dy^2 + dz^2, \text{ we obtain,}$$

$$v.\delta ds = d_ts\,.\,\delta ds = dx\,.\,\delta d_tx + dy\,.\,\delta d_ty + dz\,.\,\delta d_tz;$$

adding this to equation (1) we have,

$$\delta\,(vds) = d.\,(d_tx\,.\,\delta x + d_ty\,.\,\delta y + d_tz\,.\,\delta z),$$

$$\therefore\ \delta\int(mvds) = d_tx\,.\,\delta x + d_ty\,.\,\delta y + d_tz\,.\,\delta z + \text{constant.}$$

But since the extremities are fixed, $\therefore$ $\delta x, \delta y, \delta z$ are evanescent, and constant $= 0$,

$$\therefore\ \delta\int(mvds) = 0,$$

and $\therefore$ $\int(mvds)$, which is equal to $\int_t(mv^2)$, is a minimum.

R 85. To determine the path of a particle when acted on by forces in parallel lines.

Suppose the forces to act in lines ∥ to axis of y, and Y to be the sum of the accelerating forces which act on the body at the point xy. Suppose the body to be moving in the plane of xy; then its whole motion will lie in that plane, because there is no force acting out of it.

$$\therefore\ d_t^2x = 0,\ d_t^2y = \mathrm{Y}.$$

By integrating the first equation, we have $d_t x = c$, and by means of this eliminating t from the second, we obtain $c^2\, d_x^2y = \mathrm{Y}$, which is the differential equation of the path.

$d_x y =$ [illegible] $= \frac{d_t x \cdot d_t y}{d_t}$

86. Cor. If the path be given to find the force acting in ∥ lines by which it may be described, we have $\mathrm{Y} = c^2 d_x^2y \propto d_x^2y$.

Ex. To find the force acting in ∥ lines by which any conic section may be described. (Newton, sect. ii. schol. to prop. 8).

The general equation to a conic section is $y^2 = 2mx + nx^2$. and $\therefore\ y d_x y = m + nx$, and differentiating again, $y d_x^2 y + d_x y^2 = n$,

$$\therefore\ d_x^2y = -\frac{m^2}{y^3} \propto -\frac{1}{y^3}.$$

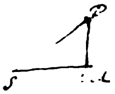
$$= \frac{ny^2 - m^2 - n(2mx + nx^2)}{y^3}$$

CENTRAL FORCES.

87. A particle is acted on by a force tending to a fixed centre; to determine its path.

The motion will take place in one plane passing through the centre of force; since there is no force to draw the particle out of the plane in which it is at any instant moving.

Take the centre of force for the origin of co-ordinates, and the plane of motion for that of xy; r = distance of the particle from the centre, xy its co-ordinates, and F the accelerating force upon it; $u = \frac{1}{r}$, $\theta = \angle$ which r makes with the prime radius, p = the $\perp$ from the origin upon a tangent to the orbit at point xy, s = length of the path described.



$$\therefore d_t^2x = -\mathrm{F}\frac{x}{r},\ d_t^2y = -\mathrm{F}\frac{y}{r} \ldots\ldots \text{(Art. 80).}$$

$$\therefore 0 = yd_t^2x - xd_t^2y, \text{ by eliminating F,}$$

$$\therefore h = yd_tx - xd_ty = pd_ts, \text{ by integration, \&c.}$$

$$\therefore d_ts^2 = \frac{h^2}{p^2} = h^2(u^2 + d_\theta u^2). = d_tx^2 + d_ty^2$$

$$\text{But } d_ts \,.\, d_t^2s = d_tx \,.\, d_t^2x + d_ty \,.\, d_t^2y = -\mathrm{F}d_tr = \frac{\mathrm{F}}{u^2}.\ d_tu,$$

$$\therefore \frac{\mathrm{F}}{u^2}.\ d_tu = \tfrac{1}{2}\,.\, d_t.(d_ts)^2 = h^2(u + d_\theta^2u)\, d_tu,$$

$$\therefore d_\theta^2u + u = \frac{\mathrm{F}}{h^2u^2}.$$

88. Cor. 1. The area described by the radius vector about the centre of force $\propto$ time. (Newton, sect. ii. prop. 1.)

For $h = yd_tx - xd_ty = r^2d_t\theta$, $\therefore ht = \int_\theta r^2 = 2.$ area.

$\therefore$ area $\propto$ time.

89. Cor. 2. If the area described about any point varies as the time, the force tends to that point. (Newton, sect. ii. prop. 2.)

For $t \propto \int_\theta r^2$, $\therefore 1 \propto r^2d_t\theta \propto yd_tx - xd_ty$; and, differentiating again, $0 = yd_t^2x - xd_t^2y = y\mathrm{X} - x\mathrm{Y}$; $\therefore \dfrac{\mathrm{Y}}{\mathrm{X}} = \dfrac{y}{x}$, that is, the force tends to the origin of co-ordinates.

90. Cor. 3. Having given the curve to find the force tending to a given point by which it may be described. (Newton, prop. 6. cors.)

By multiplying the original equations respectively by d_ty, d_tx, and subtracting; we have,

$$\frac{\mathrm{F}}{r}(yd_tx - xd_ty) = d_ty \,.\, d_t^2x - d_tx \,.\, d_t^2y = \frac{d_ts^3}{rad.curv.} = \frac{d_ts^3}{r}.d_rp,$$

$$\therefore \mathrm{F}\frac{pd_ts}{r} = \frac{d_ts^3}{r}.\ d_rp,$$

$$\therefore \mathrm{F} = d_ts^2.\ \frac{d_rp}{p} = \frac{h^2}{p^3}.\ d_rp.$$

Also, from Art. 87, $\mathrm{F} = h^2u^2\,(d_\theta^2u + u)$.

$$\text{rad. curv} = \frac{u^3\,\overline{d_\theta s}^3}{u + d_\theta^2 u} = \frac{\overline{d_t s}^3}{d_ty\, d_t^2x - d_tx\, d_t^2y}$$

Ex. 1. Find the law of force tending to the pole of the equiangular spiral. (Newton, sect. ii. prop. 9).

In this spiral $p \propto r$, $\therefore d_r p$ is constant, and $F \propto \frac{1}{r^3}$.

Ex. 2. In the ellipse, to find the law of the force tending to its centre. (Newton, sect. ii. prop. 10.)

Here $\frac{1}{p^2} = \frac{a^2 + b^2 - r^2}{a^2 b^2}$, $\therefore \frac{1}{p^3} . d_p r \propto r$, $\therefore F \propto r$.

Ex. 3. To find the law of force tending to the focus of the ellipse. (Newton, sect. iii. prop. 11.)

$$\text{Here } u = \frac{1}{r} = \frac{1 + e\cos\theta}{a(1-e^2)}, \therefore d_\theta^2 u = -\frac{e\cos\theta}{a(1-e^2)},$$

$$\therefore d_\theta^2 u + u = \text{constant},$$

$$\therefore F \propto u^2 \propto \frac{1}{r^2}.$$

91. Cor. 4. The velocity $\propto$ inversely as the perpendicular on the tangent. (Newton, prop. 1. cor. 1).

For, velocity $= d_t s = \frac{h}{p} \propto \frac{1}{p}$.

92. Cor. 5. The velocity is that due to ¼ chord of curvature. (Newton, prop. 4. cor. 12).

For, $(\text{vel.})^2 = d_t s^2 = F p d_p r = 2F . \frac{p d_p r}{2} = 2F .$ ¼ chord of curvature.

Hence, (Art. 53) if a body fall from rest, acted on by the force F continued uniform, through the space (¼ ch. curv.) it will acquire the velocity of the body in the orbit.

θ 93. Cor. 6. If $F, F_1, F_2, \ldots$ be forces, situated in different fixed centres in the plane of motion, and $\chi, \chi_1, \chi_2, \ldots$ be the chords of curvature passing through these centres, then the $(\text{vel.})^2 = \frac{1}{2} \Sigma (F\chi)$.

For $d_t^2 x = -\Sigma\left(F \frac{x}{r}\right)$, $d_t^2 y = -\Sigma\left(F \frac{y}{r}\right)$, the origin of co-ordinates being any fixed point in the plane of the forces, and $r, r_1, r_2, \ldots$ being the distances of the particle from the centres of force; $p, p_1, p_2, \ldots$ the perpendiculars from the same points upon the tangent to the curve.

$$\therefore \Sigma\frac{\mathrm{F}}{r}(yd_tx - xd_ty) = d_ty.d_t^2x - d_tx.d_t^2y = \frac{d_ts^3}{rad.curv.} = \frac{d_ts^3}{\rho},$$

$$\therefore \Sigma\left(\mathrm{F}\frac{pd_ts}{r}\right) = \frac{d_ts^3}{\rho},$$

$$\text{Now, } \rho = rd_pr = r_1d_{p_1}r_1 = r_2d_{p_2}r_2 = \ldots.$$

$$\therefore (\text{vel.})^2 = d_ts^2 = \Sigma\left(\mathrm{F}\frac{p\rho}{r}\right) = \Sigma(\mathrm{F}.\, pd_pr),$$

$$= \tfrac{1}{2}\Sigma(\mathrm{F}\chi).$$

94. The following examples are given to illustrate the general formula of Art. 87.

Ex. 1. To find the orbit described when the force $\propto$ distance.

Let μ denote the absolute accelerating force;

$$\therefore \mathrm{F} = \mu r = \frac{\mu}{u}.$$

$$\therefore d_\theta^2u + u = \frac{\mu}{h^2u^3}.$$

Multiply by $2d_\theta u$, and integrate,

$$\therefore \frac{1}{p^2} = d_\theta u^2 + u^2 = \frac{2}{\mathrm{C}} - \frac{\mu}{h^2}\cdot\frac{1}{u^2} = \frac{2}{\mathrm{C}} - \frac{\mu}{h^2}\cdot r^2 \ldots\ldots(1).$$

Now the equation of an ellipse about the centre is,

$$\frac{1}{p^2} = \frac{a^2 + b^2 - r^2}{a^2b^2}.$$

Hence the orbit is an ellipse, the centre of force being in its centre; and $h = ab\sqrt{\mu}$.

The same result may be obtained otherwise. Thus,

$$\text{From equation (1), } 2d_u\theta = \frac{2u}{\sqrt{\frac{1}{\mathrm{C}^2} - \frac{\mu}{h^2} - \left(\frac{1}{\mathrm{C}} - u^2\right)^2}},$$

$$\therefore 2(\theta - \beta) = \cos^{-1}\frac{1 - \mathrm{C}u^2}{\sqrt{1 - \frac{\mathrm{C}^2\mu}{h^2}}},$$

$$\therefore\ C = r^2 - r^2\sqrt{1 - \frac{C^2\mu}{h^2}}.\cos 2\,(\theta - \beta).$$

Let now the axis of x make an $\angle\ \beta$ with the prime radius; then,

$$C = r^2 - \sqrt{1 - \frac{C^2\mu}{h^2}}.\ r^2\,\{\cos^2(\theta-\beta) - \sin^2(\theta-\beta)\},$$

$$= x^2 + y^2 - \sqrt{1 - \frac{C^2\mu}{h^2}}.\,(x^2 - y^2),$$

$$= \left(1 - \sqrt{1 - \frac{C^2\mu}{h^2}}\right).\,x^2 + \left(1 + \sqrt{1 - \frac{C^2\mu}{h^2}}\right).\,y^2,$$

which is the equation of an ellipse about the centre.

COR. 1. The major axis makes an $\angle\,\beta$ with the prime radius.

COR. 2. If the force be repulsive, μ is negative, and the equation of the orbit becomes

$$C = \left(\sqrt{1 + \frac{C^2\mu}{h^2}} + 1\right).\,y^2 - \left(\sqrt{1 + \frac{C^2\mu}{h^2}} - 1\right).\,x^2,$$ which is that of an hyperbola about the centre.

COR. 3. The periodic time $= \frac{2\pi ab}{h} = \frac{2\pi\,ab}{ab\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}}$, which is the same for all ellipses. (Newton, sect. ii. prop. 10. cor. 2.)

95. A body is projected from a given point, with a given velocity, and in a given direction, about a centre of force varying as the distance; to find the magnitude and position of the ellipse described. (Newton, sect. ii. prop. 10. cor. 1.)

Let the prime radius pass through the point of projection; and suppose the body projected with a velocity v, from a distance c, in a direction making an $\angle\ \alpha$ with the radius vector.

Then,

$$\because\ \frac{1}{p^2} = \frac{2}{C} - \frac{\mu}{h^2}.\,r^2 = \frac{a^2+b^2}{a^2\,b^2} - \frac{r^2}{a^2\,b^2}.$$

$$\therefore\ \frac{1}{c^2\sin^2\alpha} = \frac{a^2+b^2}{a^2 b^2} - \frac{c^2}{a^2\,b^2},\ \text{and } v = \frac{h}{c\sin\alpha} = \frac{ab\sqrt{\mu}}{c\sin\alpha}.\ \text{(Art. 88)}$$

[illegible]

$$\therefore\ a^2 + b^2 = c^2 + \frac{a^2\, b^2}{c^2 \sin^2 \alpha} = c^2 + \frac{v^2}{\mu} \ .\ .\ .\ .\ .\ (1),$$

$$2ab = \frac{2vc}{\sqrt{\mu}} \,.\, \sin \alpha.$$

The two last equations determine the magnitude of the ellipse.

And, because $\cos 2\beta = \dfrac{1 - \dfrac{\mathrm{C}}{c^2}}{\sqrt{1 - \dfrac{\mathrm{C}^2 \mu}{h^2}}}$, by Art. 94,

$\therefore\ \cot 2\beta = \cot 2\alpha + \dfrac{c^2 \mu}{v^2} . \operatorname{cosec} 2\alpha$, by eliminating C and h; from which the position of the major axis (Art. 94, Cor. 1) is determined.

Cor. 1. From equation (1) it appears that the length of the semi-diameter, which is conjugate to c, is $\dfrac{v}{\sqrt{\mu}}$, and $\therefore$ the velocity at any point in an ellipse varies as the corresponding conjugate diameter.

Cor. 2. The equation of position may be reduced to the form $\sin 2(\alpha - \beta) = \dfrac{c^2 \mu}{v^2} . \sin 2\beta$; hence the $\angle\ 2\beta$ will be determined by dividing the $\angle\ 2\alpha$ into two parts whose sines are in the proportion of $c^2 : \dfrac{v^2}{\mu}$.

96. Ex. 2. To find the orbit when the force $\propto \dfrac{1}{\mathrm{D}^2}$.

Here $d_\theta^{\,2} u + u = \dfrac{\mu}{h^2}$, which is a linear equation of the 2nd order,

$$\therefore\ \frac{1}{r} = u = \frac{\mu}{h^2} \{1 + e \cos(\theta - \beta)\} ;$$

hence the orbit is a conic section about the focus, whose latus rectum $= \dfrac{2h^2}{\mu}$ (Newton, sect. iii. prop. 14), and eccentricity $= e$.

COR. Since $\frac{h^2}{\mu} = \frac{b^2}{a}$, $\therefore$ periodic time in an ellipse $= \frac{2\pi ab}{h}$ $= \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$, which is the same for all ellipses having the same axis major. (Newton, sect. iii. prop. 15.)

97. A body is projected from a given point, with a given velocity, and in a given direction, about a centre of force $\propto \frac{1}{D^2}$; to find the conic section described. (Newton, sect. iii. prop. 13. cor. 3.)

$$d_\theta^2 u + u = \frac{\mu}{h^2}.$$

Multiply by $2d_\theta u$, and integrate,

$$\therefore \; d_\theta u^2 + u^2 = \frac{2\mu}{h^2} \cdot u + \frac{C}{h^2},$$

$$\therefore \; (\text{vel.})^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C,$$

$$\therefore \; v^2 = \frac{h^2}{c^2 \sin^2 \alpha} = \frac{2\mu}{c} + C, \text{ using the notation of Art. 95,}$$

$$\therefore \; h = cv \sin \alpha, \text{ and } C = v^2 - \frac{2\mu}{c}.$$

$$\therefore \; \frac{c^2 v^2 \sin^2 \alpha}{p^2} = \frac{2\mu}{r} + v^2 - \frac{2\mu}{c},$$

$$\text{or, } \frac{v^2}{2\mu} \cdot \frac{c^2 \sin^2 \alpha}{p^2} = \frac{1}{r} + \frac{v^2}{2\mu} - \frac{1}{c}.$$

Now the general equation of a conic section, about the focus,

$b^2 \frac{r}{2a-r} = \frac{SP}{HP}$

is $\frac{b^2}{2ap^2} = \frac{1}{r} \mp \frac{1}{2a}$,

$$\therefore \frac{1}{2a} = \frac{v^2}{2\mu} \sim \frac{1}{c}, \text{ and } \frac{b^2}{a} = \frac{v^2}{\mu} \cdot c^2 \sin^2 \alpha = \tfrac{1}{2} \text{ the lat. rect.}$$

The orbit will be an ellipse, parabola, or hyperbola, according as v^2 is $>$, $=$, or $<$ $\frac{2\mu}{c}$, and the two last equations will determine its magnitude.

Again, in the equation of (Art. 96), $\frac{1}{r} = \frac{\mu}{h^2} + \frac{\mu e}{h^2}.\cos(\theta-\beta)$, let the prime radius pass through the point of projection.

$$\therefore \frac{1}{c} = \frac{\mu}{h^2} + \frac{\mu e}{h^2}.\cos\beta.$$

But $\frac{d_\theta r}{r^2} = \frac{\mu e}{h^2}.\sin(\theta-\beta)$, $\therefore\ \frac{1}{c}.\cot\alpha = \frac{\mu e}{h^2}.\sin\beta$,

$$\therefore \cot\alpha.\cot\beta = 1 - \frac{\mu c}{h^2}, \text{ by eliminating } e;$$

$$= 1 - \frac{\mu}{cv^2\sin^2\alpha}.$$

$$\therefore \cot\beta = \tan\alpha - \frac{2\mu}{cv^2}.\operatorname{cosec}2\alpha,$$

which determines the position of the major axis.

Cor. 1. Suppose the velocity is n times that acquired from infinity.

$$\therefore v^2 = n^2.\frac{2\mu}{c},\ \therefore \frac{1}{2a} = \frac{1\sim n^2}{c},\ \therefore \text{the major axis} = \frac{c}{1\sim n^2},$$

$$\text{and } \frac{b^2}{a^2} = \frac{2c^2n^2\sin^2\alpha}{c}.\frac{2(1\sim n^2)}{c} = 4n^2(1\sim n^2)\sin^2\alpha,$$

$$\therefore \text{the eccentricity} = \sqrt{1-4n^2(1\sim n^2)\sin^2\alpha}.$$

$$\text{and the minor axis} = 2b = \frac{2nc\sin\alpha}{\sqrt{1\sim n^2}}.$$

Hence the major axis is independent of the direction; and the eccentricity of the distance of projection.

The demonstrations of the two following problems are omitted because they contain no new difficulty.

A number of bodies are projected from a given point in all directions, with the same velocity (v), and acted on by a force tending to a fixed centre varying the distance; to shew that the surface which touches all the orbits described is a sphere whose radius $= \frac{v}{\sqrt{\mu}}$; its centre being in the centre of force.

The same things being supposed, except that the force varies as $\frac{1}{D^2}$; the surface will be that generated by the revolution of the conic section (about a line through the point of projection and centre of force) whose equation is,

$$\frac{c}{r} = \frac{\mu}{cv^2} - \frac{cv^2}{4\mu} + \left(1 - \frac{\mu}{cv^2} - \frac{cv^2}{4\mu}\right) . \cos\theta.$$

98. To find the time of describing any portion of an elliptic orbit, force $\propto \frac{1}{D^2}$. (Newton, sect. vi. prop. 31.)

The reader will find this question fully discussed in Airy's Tracts. The following is however sufficiently curious and useful to warrant its insertion.

Let P (Fig. 14) be the place after the time t from leaving the apse A. S its focus, AQM the circumscribing circle, C its centre; QPN $\perp$ AM, join SP, SQ, CQ. $v =$ ASP, $u =$ ACQ, which are called the *true* and *eccentric anomalies;* $m =$ the $\angle$ described in 1″ by a body, revolving uniformly in a circle, in the same periodic time as in the ellipse. Then, by Cor. to Art. 96, $\frac{\sqrt{\mu}}{a^{\frac{3}{2}}} . t$ $(= mt)$ is the $\angle$ uniformly described in the time t, which is called the *mean anomaly;* and since $t \propto$ area ASP $\propto$ area ASQ,

$$\therefore \frac{mt}{2\pi} = \frac{\text{area ASQ}}{\text{area of circle}} = \frac{\text{area ASQ}}{a^2\pi},$$

$$\therefore \tfrac{1}{2} a^2 mt = \text{area ASQ} = \text{area ACQ} - \triangle\ \text{SCQ},$$

$$= \tfrac{1}{2} a^2 (u - e \sin u).$$

$$\therefore mt = u - e \sin u \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Again,

$$a \cos u = \text{CS} + \text{SN} = ae + \frac{a(1-e^2)\cos v}{1 + e\cos v} = a . \frac{e + \cos v}{1 + e\cos v}$$

$$\therefore \frac{1 - \cos u}{1 + \cos u} = \frac{1 - e}{1 + e} . \frac{1 - \cos v}{1 + \cos v},$$

$$\therefore \tan\frac{u}{2} = \left(\frac{1-e}{1+e}\right)^{\frac{1}{2}} . \tan\frac{v}{2} \quad . \quad . \quad . \quad (2).$$

From equation (2) we can find u, when v is known; and then (1) will give t.

99. To find the time of describing any portion of a parabolic orbit, force $\propto \frac{1}{\mathrm{D}^2}$. (Newton, sect. vi. prop. 30.)

Let $4a =$ the latus rectum $= \frac{2h^2}{\mu}$ (Art. 96); and the equation of the orbit is $r = a \sec^2 \frac{\theta}{2}$.

$$\therefore\ t = \int_\theta \frac{r^2}{h} \text{ (Art.88)} = \frac{a^{\frac{3}{2}}}{\sqrt{2\mu}} \cdot \int_\theta \sec^4 \frac{\theta}{2} = \frac{2a^{\frac{3}{2}}}{\sqrt{2\mu}} \int_{\tan\frac{\theta}{2}} \left(1 + \tan^2 \frac{\theta}{2}\right).$$

$$= \frac{a^{\frac{3}{2}}}{3} \sqrt{\frac{2}{\mu}} \cdot \left(3 \tan \frac{\theta}{2} + \tan^3 \frac{\theta}{2}\right) + \mathrm{C}.$$

100. Ex. 3. To find the orbit, when the force $\propto \frac{1}{\mathrm{D}^3}$.

Here $\mathrm{F} = \mu u^3$, and $\therefore\ d_\theta^2 u + u - \frac{\mu}{h^2} u = 0$.

Multiply by $2d_\theta u$, and integrate, $\therefore\ d_\theta u^2 + u^2 - \frac{\mu}{h^2} . u^2 = \mp \mathrm{C}$, which may be reduced to the form

$\frac{a^2}{r^2} - \frac{b^2}{p^2} = \pm 1$, by making $\frac{\mu}{\mathrm{C}h^2} = a^2$, and $\frac{1}{\mathrm{C}} = b^2$.

$$\text{Hence } \left(\frac{a^2}{b^2} - 1\right)^{\frac{1}{2}} = \frac{\sqrt{a^2 - b^2}.\ d_\theta u}{\sqrt{(a^2 - b^2) u^2 \mp 1}},$$

$$\therefore\ e^{(\theta - \beta) \cdot \sqrt{\frac{a^2}{b^2} - 1}} = u \sqrt{a^2 - b^2} + \sqrt{(a^2 - b^2) u^2 \mp 1}$$

$$\text{and } \pm\, e^{-(\theta - \beta) \cdot \sqrt{\frac{a^2}{b^2} - 1}} = u \sqrt{a^2 - b^2} - \sqrt{(a^2 - b^2) u^2 \mp 1}.$$

Now we may take the prime radius in such a position, that $\beta = 0$, and then we shall have for determining the orbit, either of these equations,

$$\frac{a^2}{r^2} - \frac{b^2}{p^2} = \pm 1; \quad \frac{2\sqrt{a^2 - b^2}}{r} = e^{\theta \sqrt{\frac{a^2}{b^2} - 1}} \pm e^{-\theta \sqrt{\frac{a^2}{b^2} - 1}}$$

Case 1. When $a > b$, and the upper sign is used, the equations become

$$\frac{a^2}{r^2} - \frac{b^2}{p^2} = 1, \frac{2\sqrt{a^2 - b^2}}{r} = e^{\theta\sqrt{\frac{a^2}{b^2}-1}} + e^{-\theta\sqrt{\frac{a^2}{b^2}-1}}.$$

Let SA (Fig. 15) be the prime radius, then when $\theta = 0$, $SA = \sqrt{a^2 - b^2}$, and $p = SA$, $\therefore$ the curve cuts SA perpendicularly in A; and as θ increases, r diminishes gradually, till $\theta = \infty$, when $r = 0$. And as the equation is not changed by writing $-\theta$ for θ, the curve consists of two similar and equal parts, on contrary sides of SA, winding round the centre S, and approaching continually nearer and nearer, but never actually reaching it.

Case 2. When a and b are both ∞, the former equation becomes $p = \frac{b}{a} \cdot r = Cr$, which belongs to the equiangular spiral.

Case 3. When $a > b$, and the lower sign is used, the equations become

$$\frac{b^2}{p^2} - \frac{a^2}{r^2} = 1; \frac{2\sqrt{a^2 - b^2}}{r} = e^{\theta\sqrt{\frac{a^2}{b^2}-1}} - e^{-\theta\sqrt{\frac{a^2}{b^2}-1}}.$$

Take SA (Fig. 16) for the prime radius; then when $\theta = 0$, $r = \infty$, and $p = b$; $\therefore$ draw SY $\perp$ SA and $= b$, and YZ $\parallel$ SA will be an asymptote.

Now as θ increases, r diminishes gradually until $\theta = \infty$, when $r = 0$. The curve $\therefore$ having the asymptote YZ winds continually about the centre S, approaching continually nearer and nearer, but never actually reaching it. The complete curve consists of two such orbits, situated as in the Figure.

Case 4. When $a = b$, $\frac{1}{p^2} - \frac{1}{r^2} = \frac{1}{a^2}$, which is the equation of the hyperbolic spiral, the form of which is similar to Fig. 16.

Case 5. When $a < b$, $\frac{b^2}{p^2} - \frac{a^2}{r^2} = 1$;

and $$\frac{2\sqrt{b^2-a^2}.\sqrt{-1}}{r} = e^{\theta\sqrt{1-\frac{a^2}{b^2}}.\sqrt{-1}} - e^{-\theta\sqrt{1-\frac{a^2}{b^2}}.\sqrt{-1}},$$

$$= 2\sqrt{-1}.\ \sin\left\{\theta\left(1-\frac{a^2}{b^2}\right)^{\frac{1}{2}}\right\}$$

$$\therefore\ \frac{\sqrt{b^2-a^2}}{r} = \sin\left\{\theta\left(1-\frac{a^2}{b^2}\right)^{\frac{1}{2}}\right\}.$$

Here when $\theta = 0$, $r = \infty$, and $p = b$, $\therefore$ there is an asymptote as in the 3rd Case; and as θ increases, r diminishes until $\theta\left(1-\frac{a^2}{b^2}\right)^{\frac{1}{2}} = \frac{\pi}{2}$, when $r = \sqrt{b^2-a^2}$; after this, as θ increases, r increases in the same manner as it before diminished, until $\theta\left(1-\frac{a^2}{b^2}\right)^{\frac{1}{2}} = \pi$, when $r = \infty$, and $p = b$. Hence drawing SB (Fig. 17) making an $\angle$ with SA determined from $\theta\left(1-\frac{a^2}{b^2}\right)^{\frac{1}{2}} = \frac{\pi}{2}$, the parts of the curve will be similar and equal on contrary sides of BSB′.

The complete curve consists of an ∞ number of orbits, similar to the one in the Figure, but differently situated; but, both in this case, and Cases 3 and 4, it is quite evident the complete curve cannot be described by a revolving body.

Case 6. When the force is repulsive, μ is negative, and $\because \frac{a^2}{b^2} = \frac{-\mu}{h^2}$, $\therefore$ the equations become $\frac{a^2}{r^2} + \frac{b^2}{p^2} = 1$,

and $$\frac{\sqrt{a^2+b^2}}{r} = \sin\left\{\theta\left(1+\frac{a^2}{b^2}\right)^{\frac{1}{2}}\right\}.$$

This is the same species of curve as that in Case 5; but, on account of $\left(1+\frac{a^2}{b^2}\right)^{\frac{1}{2}}$ being greater than unity, its form will be that of Fig. 18.

These curves are known by the name of Cotes' spirals.

Cor. Since there is an apse when $d_\theta r = 0$, it would appear that the $\angle$ between two apses, called the apsidal $\angle$, in Case 5, $= \frac{\pi}{\left(1 - \frac{a^2}{b^2}\right)^{\frac{1}{2}}}$; but from what has just been said, it appears there is in reality no aspidal angle.

The time in any of these curves may be found from the equation $t = \int_\theta \cdot \frac{r^2}{h}$ (Art. 88.) $= \frac{a}{b} \cdot \int_\theta \frac{r^2}{\sqrt{\mu}}$.

101. Ex. 4. To determine the orbit, when force $\propto \frac{1}{D^5}$.

$$\text{Here } F = \mu u^5, \therefore d_\theta^2 u + u = \frac{\mu}{h^2} \cdot u^3.$$

Multiply by $2 d_\theta u$, and integrate,

$$\therefore d_\theta u^2 + u^2 = \frac{\mu}{2h^2} \cdot u^4 + C.$$

This equation cannot be generally integrated in finite terms. In certain cases, however, the integration can be effected; as, for instance, when $C = \frac{h^2}{\mu}$; and then

$$d_\theta u = \sqrt{\frac{\mu}{2h^2}} \cdot \left(\frac{h^2}{\mu} - u^2\right)$$

which, by integration, gives $r = \frac{\sqrt{\mu}}{h} \cdot \frac{e^{\theta\sqrt{2}} \pm C}{e^{\theta\sqrt{2}} \mp C}$.

Now taking the prime radius in such a position that $C = 1$, and putting $\frac{\sqrt{\mu}}{h} = a$, we have

Case 1. $\frac{r}{a} = \frac{e^{\theta\sqrt{2}} + 1}{e^{\theta\sqrt{2}} - 1}$, by using the top sign.

Now when $\theta = 0$, $r = \infty$, and as θ increases, r decreases gradually, until $\theta = \infty$, when $r = a$. Hence the body, coming from an ∞ distance, approaches continually nearer and nearer to the circumference of a circle whose radius is $a = SA$ (Fig. 19), without ever actually reaching it. Hence the circle whose radius is SA is called an *asymptotic circle.*

Case 2. $\frac{r}{a} = \frac{e^{\theta\sqrt{2}} - 1}{e^{\theta\sqrt{2}} + 1}$, by using the lower sign.

Here when $\theta = 0$, $r = 0$; and r increases gradually, as θ increases, until $\theta = \infty$, when $r = a$. The orbit in this case lies within the asymptotic circle. (Fig. 20).

102. Ex. 5. To find the orbit when the force $\propto \frac{1}{\text{D}^n}$, and the velocity is equal to that acquired by a body in falling from an infinite distance.

The (velocity)^2 acquired from an infinite distance $= 2\mu . \frac{u^{n-1}}{n-1}$.

$$\text{Now } d_\theta{}^2 u + u = \frac{\mu}{h^2} \cdot u^{n-2},$$

$$\therefore\ d_\theta u^2 + u^2 = \frac{2\mu}{h^2} \cdot \frac{u^{n-1}}{n-1}, \because \text{ const.} = 0.$$

$$\therefore\ 1 = \frac{-r^{\frac{n-5}{2}} d_\theta r}{\sqrt{\frac{2\mu}{(n-1)h^2} - r^{n-3}}},$$

and by integrating, and supposing the prime radius in such a position that const. $= 0$, and writing $a^{\frac{3-n}{2}}$ for $h\sqrt{\frac{n-1}{2\mu}}$, we obtain

$$\left(\frac{r}{a}\right)^{\frac{n-3}{2}} = \cos\left(\frac{n-3}{2}\theta\right) \ldots\ldots (1).$$

or, $\because \frac{1}{p^2} = d_\theta u^2 + u^2$, we obtain $r^{n-1} = a^{n-3}\, p^2$, either of which equations may be used in determining the form of the curve.

Case 1. When n is > 3.

Here $r = a$, and $p = a$, when $\theta = 0$; and r diminishes gradually as θ increases until $\theta = \frac{\pi}{n-3}$, when $r = 0$; and if $\frac{n-3}{2}$ be an even number, this branch of the curve will terminate at the origin; but if it be an odd number, it will pass through the origin, and describe a branch similar to the last; and as the equation (1) is not altered by writing $-\theta$, for θ, the curve will consist of similar parts on the contrary sides of the prime radius SA. Figs. 21 and 22 correspond to these two curves. The apsidal $\angle$* in the latter curve $= \frac{n-1}{n-3} \cdot \pi$, the body not being supposed at an apse when passing through the centre.

The orbit (Fig. 22) will always return into itself, the number of nodes being $2x \cdot \frac{n-3}{n-1}$, where x is the least whole number that will render the expression an integer.

CASE 2. When n is < 3, and > 1, the equations become,

$$\left(\frac{r}{a}\right)^{\frac{3-n}{2}} = \sec\left(\frac{3-n}{2}\,\theta\right), \text{ and } p^2 = a^{3-n} r^{n-1}.$$

Here $r = a$, and $p = a$, when $\theta = 0$; and as θ increases, r increases, until $\theta = \frac{\pi}{3-n}$ when $r = \infty$, and $p = \infty$.

Hence there is no asymptote; but since the equation of the curve is not altered by writing $-\theta$, for θ, it will consist of similar parts on the contrary sides of the prime radius SA. (Fig. 23).

CASE 3. When n is < 1, the latter equation becomes $r^{1-n} p^2 = a^{3-n}$.

This curve differs from that in Case 2, only in having asymptotes ($\because$ when $r = \infty, p = 0$) passing through the origin S, and making angles with SA $= \frac{\pi}{3-n}$, (Fig. 24). The force is repulsive in this species.

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* The apsidal angle is the angle between two successive apses.

103. The characteristic property of all these curves, is this; that the $\angle$ PSY $= \frac{n \sim 3}{2}$. ASP; SY being a perpendicular on the tangent PY at any point P in the curve.

$$\text{For } \cos\left(\frac{n \sim 3}{2}\theta\right) = \left(\frac{r}{a}\right)^{\frac{n-3}{n}} = \frac{p}{r} = \cos \text{PSY}.$$

$$\therefore \text{PSY} = \frac{n \sim 3}{2}.\ \text{PSA}.$$

When $n = 2$, PSY $= \frac{\text{PSA}}{2}$, or PSY $=$ YSA, (Fig. 23) which is the property of the parabola.

Cor. After leaving an apse, the body will either fall to the centre or fly off *in infinitum* when it has described an $\angle\ \frac{\pi}{n \sim 3}$.

104. To find the condition that a body may describe an orbit having an asymptotic circle, about a centre of force $\propto \frac{1}{\text{D}^n}$.

$$\text{Here F} = \mu u^n,\ \therefore\ d_\theta{}^2 u + u - \frac{\mu}{h^2} . u^{n-2} = 0,$$

$$\therefore\ d_\theta u^2 = \text{C} - u^2 + \frac{2\mu}{h^2} \cdot \frac{u^{n-1}}{n-1}.$$

But if the orbit have an asymptotic circle, (let $\frac{1}{c}$ be its radius) it will have a contact of an ∞ order with it; and $d_\theta u = 0$, $d_\theta{}^2 u = 0$, $d_\theta{}^3 u = 0$, &c. when $u = c$.

$$\left.\begin{aligned} \therefore\ 0 &= \text{C} - c^2 + \frac{2\mu}{h^2} \cdot \frac{c^{n-1}}{n-1} \\ 0 &= \quad\ \ - c \ + \frac{\mu}{h^2} \cdot c^{n-2} \end{aligned}\right\}\ \therefore\ \text{C} = \frac{n-3}{n-1} . c^2, \text{ and } \frac{h^2}{\mu} = c^{n-3}.$$

Let the vel. in the orbit $= \epsilon$ times that in a circle at the same distance, and $\frac{1}{b} =$ distance of the point of projection from the centre, $\beta = \angle$ which the direction of projection makes with the distance.

$\therefore\ \mu c^{n-3} = h^2 = (\text{vel.})^2 . (\text{perpend.})^2$ on direction of projection

$$= \varepsilon^2 . \mu b^{n-1} . \frac{\sin^2\beta}{b^2},\ \therefore \left(\frac{c}{b}\right)^{n-3} = \varepsilon^2 . \sin^2\beta,$$

and $\varepsilon^2 . \mu b^{n-1} = (\text{vel.})^2 = (d_\theta u^2 + u^2)_b\, h^2 = Ch^2 + 2\mu . \dfrac{b^{n-1}}{n-1}$

$$\therefore \left(\varepsilon^2 - \frac{2}{n-1}\right) . b^{n-1} = \frac{Ch^2}{\mu} = \frac{n-3}{n-1} . c^{n-1},\ \therefore \left(\frac{c}{b}\right)^{n-1} = \frac{\varepsilon^2 . (n-1) - 2}{n-3},$$

and $\therefore \sin\beta = \dfrac{1}{\varepsilon} . \left(\dfrac{c}{b}\right)^{\frac{n-3}{2}} = \dfrac{1}{\varepsilon} . \left(\dfrac{\overline{n-1} . \varepsilon^2 - 2}{n-3}\right)^{\frac{n-3}{2n-2}}.$

Cor 1. Hence the radius of the asymptotic circle

$$= \frac{1}{c} = \frac{1}{b} . \left(\frac{n-3}{\varepsilon^2 (n-1) - 2}\right)^{\frac{1}{n-1}},$$

and it will be interior or exterior to the orbit according as $c >$ or $< b$,

or, $\varepsilon^2 . (n-1) - 2 > < n - 3$,

or, $\varepsilon^2 > < 1$, when n is > 1;

or, $\varepsilon^2 < > 1$, when n is < 1.

105. Ex. 6. To determine the orbit, when the force

$$\propto \frac{\mu}{D^2} + \frac{\mu'}{D^3}.$$

Here $d_\theta^2 u + u = \dfrac{\mu}{h^2} + \dfrac{\mu'}{h^2} . u$,

$$\therefore\ d_\theta^2 u + \left(1 - \frac{\mu'}{h^2}\right) u = \frac{\mu}{h^2},$$

$$\text{or, } d_\theta^2\left(u + \frac{\mu}{\mu' - h^2}\right) + \left(1 - \frac{\mu'}{h^2}\right)\left(u + \frac{\mu}{\mu' - h^2}\right) = 0.$$

Hence, Case 1.

$$u = \frac{\mu}{h^2 - \mu'}\left\{1 + e\cos\theta\left(1 - \frac{\mu'}{h^2}\right)^{\frac{1}{2}}\right\},\ \text{when } h^2 > \mu'.$$

or, $\dfrac{a}{r} = 1 + e\cos c\theta$; by making $h^2 - \mu' = \mu a = c^2 h^2$.

Now this equation would coincide with that of a conic section, if we had $\cos\theta$ instead of $\cos c\theta$. From this property we derive the following construction :—

Let AP (Fig. 25) be the conic section whose equation is $\frac{a}{r} = 1 + e \cos \theta$, and let a body P revolve in this orbit acted on by a force tending to the centre S, as in Ex. 2; $c\theta = \angle$ ASP.

Suppose also the orbit itself to revolve (carrying the body along with it) with an angular velocity which : the angular velocity of P in the orbit :: $1-c : c$; now ap and p being contemporaneous positions of the orbit and body revolving in it, the $\angle$ PSp = ASa = $(1-c)\,\theta$; $= \theta - c\theta = \theta - \text{ASP}$, $\therefore$ ASp = θ; and, by the equation of the conic section,

$$\frac{a}{\text{S}p} = \frac{a}{\text{SP}} = 1 + e\cos \text{ASP} = 1 + e\cos c\theta = \frac{a}{r},$$

$$\therefore \text{S}p = r;$$

hence the path Ap traced out by p is the orbit whose equation is $\frac{a}{r} = 1 + e \cos c\theta$. (Newton, sect. ix. prop. 43). See also Airy's Tracts, *Lunar Theory.*

If the conic section be an ellipse, that is, if e be < 1, the body will arrive at an apse after describing an angle $= \frac{\pi}{c}$ $= \frac{\pi}{\left(1 - \frac{\mu'}{h^2}\right)^{\frac{1}{2}}}$, which is $\therefore$ the apsidal angle.

In all other cases it will never arrive at an apse after leaving A.

CASE 2. $\frac{a}{r} + \frac{a}{b} = \epsilon^{c\theta} \pm \epsilon^{-c\theta}$ (by making $\mu' - h^2 = \mu b$ $= c^2h^2$) when $h^2 < \mu'$.

Using the top sign, $r = \frac{ab}{2b - a}$ when $\theta = 0$, and as θ increases, r decreases gradually, and is 0 when θ is ∞. Hence the body after an ∞ number of revolutions falls to the centre. The form of the curve is that of Fig. 15.

Using the lower sign, $r = -b$ when $\theta = 0$, and as θ increases, r increases gradually until $\frac{a}{b} = e^{c\theta} - e^{-c\theta}$, when $r = \infty$; after this, r decreases and is $= 0$ when $\theta = \infty$.

When θ is negative, the equation is $-\frac{a}{r} = \frac{a}{b} + e^{c\theta} - e^{-c\theta}$; and on this side of SA, as θ increases, r gradually decreases until $\theta = \infty$ when $r = 0$. The form of the orbit is $\therefore$ that of Fig. 26. There is no apse $\because$ $d_\theta u = c\,(e^{c\theta} + e^{-c\theta})$, which cannot $= 0$. The body will fly off *in infinitum* after describing an $\angle\ \theta$ determined from the equation $\frac{a}{b} = e^{c\theta} - e^{-c\theta}$.

106. A body revolves in an orbit, which is very nearly circular, acted on by a force varying as any function of the distance; to find the angle between the apsides. (Newton, sect. ix. prop. 45).

Suppose $F = u^2\phi u$, $\frac{1}{c} =$ the greatest distance, $u = c + x$, where x is always very small.

$\therefore F = u^2\phi\,(c+x) = u^2(\phi c + x\,\phi' c + \&c.) = u^2\,(\phi c + x\phi' c)$ nearly; and $\because$ the orbit is very nearly circular, $\therefore$ (vel.)^2 at distance $\frac{1}{c}$ very nearly $=$ (vel.)^2 in a circle at the same distance; we may $\therefore$ assume it $= (1-m)c\phi c$, m being very small.

$\therefore\ c^2h^2 = (1-m)\,c\phi c,\ \therefore\ h^2 = (1-m)\,.\,\frac{\phi c}{c}.$

$$\text{But } d_\theta^{\,2}u + u = \frac{\phi c}{h^2} + \frac{\phi' c}{h^2}\cdot x,$$

$$= \frac{c}{\phi c}\cdot(1+m)\,(\phi c + \phi' c\,.\,x),$$

$$= c + mc + \frac{c\phi' c}{\phi c}\,.\,x,$$

$$\therefore\ d_\theta^{\,2}u + \left(1 - \frac{c\phi' c}{\phi c}\right)u = c + mc - \frac{c^2\phi' c}{\phi c},\ \because\ x = u - c,$$

and ∴ by the 1st Case of Art. 105, the apsidal angle

$$=\frac{\pi}{\left(1-\frac{c\phi'c}{\phi c}\right)^{\frac{1}{2}}}.$$

COR. If a body describe an orbit which is very nearly circular, the force may ultimately be supposed to consist of two parts; one varying inversely as the square, and the other inversely as the cube of the distance; and ∴ the orbit may be supposed to be traced out by a body revolving in an ellipse, while the ellipse itself revolves, as in the former Case of last Article; which illustrates the assumption made by Newton in the proposition referred to.

CHAPTER V.

ON THE CONSTRAINED MOTION OF A PARTICLE OF MATTER, WHEN ACTED ON BY GIVEN FORCES.

107. In considering the motion of a particle subject to some restraint, such as tension, or reaction, if we add to the forces impressed on it that which it suffers from the restraint, we may consider it as free; and accordingly we may apply the equations of free motion deduced in Art. 80. A few examples of this kind are annexed.

Ex. 1. A particle moves on a plane curve, acted on by any forces in that plane; to determine the motion.

Resolve the impressed forces || the co-ordinate axes of x, y; suppose X, Y the resolved parts; m = the mass of the particle, P = its pressure on the surface, which is equal and opposite to the reaction of the curve; x, y co-ordinates of m, v = its velocity, and s = the arc described.

$$\therefore md_t^2x = X - Pd_ty, \quad md_t^2y = Y + Pd_tx;$$

by eliminating P we obtain,

$$mdxd_t^2x + mdyd_t^2y = Xdx + Ydy,$$

$$\therefore mv^2 = C + 2\int(Xdx + Ydy) \ldots\ldots (1).$$

Again,

$$Pd_tx^2 + Pd_ty^2 = Xd_ty - Yd_tx + m\,(d_txd_t^2y - d_tyd_t^2x)$$

$$\therefore P = Xd_ty - Yd_tx + m\,.\,\frac{d_ts^2}{rad.\ curv.}.$$

$$= Xd_ty - Yd_tx + \frac{m\,.\,v^2}{\rho}.$$

= the pressure of the particle arising from the forces X, Y; + the pressure arising from the centrifugal force $\frac{v^2}{\rho}$.

108. Cor. 1. If the particle be acted on by no forces, the velocity will be uniform, as appears from equation (1), and the pressure on the curve $\left(=\frac{mv^2}{\rho}\right)$ will arise only from the motion of the particle, and being $\perp$ to the surface, always tends *from the centre of curvature;* on this account, $\frac{v^2}{\rho}$ is called the centrifugal force.

109. Cor. 2. The actual velocity about the centre of curvature, $(= v^2) = \rho^2$. (angular velocity)^2, and $\therefore$ centrifugal force = (radius of curvature) . (angular velocity)^2.

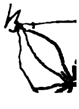
If a particle move in any curve whatever, about any fixed point, the motion may be resolved into two, one in the direction of the distance, and the other $\perp$ to it; and the latter motion would generate a centrifugal force = (distance) . (angular velocity)^2. This expression is therefore the true measure of the force which arises from angular motion, about a fixed point, whatever be the path of the body.

110. Cor. 3. If the particle be made to oscillate, or perform revolutions in a curve, by means of a string, the tension will $= \mathrm{X}d_s x - \mathrm{Y}d_s y + \frac{m \, . \, v^2}{\rho}$.

111. Cor. 4. It appears from equation (1), and the reasoning of (Art. 82), that the *vis viva* (mv^2) is independent of the form of the curve, along which the particle is constrained to move; and also of the reaction; and from (Art. 84) that its path is such that $\int(mvds)$ or $\int_t(mv^2)$ is less than if it had taken any other course.

112. Ex. 2. To find the time of descent in a cycloid, whose axis is vertical, and base horizontal.

D (Fig. 27) is the point from which the particle begins to descend; E its position after an interval t. Draw EM $\parallel$ CB,



and join AM, BM. Then, if BA be taken to represent the force of gravity, MA being || to the tangent at E will represent the force down the curve, and the part BM being ⊥ to the curve will be wholly destroyed by the reaction. a = BC, s = AE, β = AD. in propn. of Cx

$$\therefore \frac{\text{force down the curve}}{\text{force of gravity } (= g)} = \frac{2\text{MA}}{2\text{BA}} = \frac{s}{a},$$

$$\therefore \text{accel. force down the curve} = \frac{s}{a}\cdot g,$$

$$\therefore v d_s v = -\frac{g}{a}\cdot s,$$

$$\therefore v^2 = \frac{g}{a}\,(\beta^2 - s^2),$$

$$\therefore t = \sqrt{\frac{a}{g}}\cdot\int_s \frac{-1}{\sqrt{\beta^2 - s^2}} = \sqrt{\frac{a}{g}}\cdot \cos^{-1}\left(\frac{s}{\beta}\right);$$

and this taken between $s = \pm\,\beta$, gives the whole time of an oscillation $= \pi\sqrt{\frac{a}{g}}$.

113. Ex. 3. To find the time of oscillation in a circular arc; the particle being acted on by the force of gravity.

Let A (Fig. 28) be the centre, B the lowest point of the arc, C the point from which the body begins to descend, CP the arc described in time t; a = AB, h = BD, x = BM, s = BP.

$\therefore$ vel. at P $= \sqrt{2g\,(h - x)}$. (Arts. 111, 53).

$$\text{and } t = \int_x \frac{-d_x s}{\text{vel.}} = \frac{a}{\sqrt{2g}}\cdot\int_x \frac{-1}{\sqrt{2ax - x^2}\,.\,\sqrt{h - x}}$$

$$= \frac{a}{\sqrt{2g}}\cdot\int_x \frac{-1}{\sqrt{hx - x^2}\,.\,\sqrt{2a - x}},$$

$$= \tfrac{1}{2}\sqrt{\frac{a}{g}}\cdot\int_x \frac{-1}{\sqrt{hx - x^2}}\cdot\left\{1 + \frac{1}{2}\cdot\frac{x}{2a} + \frac{1.3}{2.4}\cdot\left(\frac{x}{2a}\right)^2 + \frac{1.3.5}{2.4.6}\cdot\left(\frac{x}{2a}\right)^3 + \&c.\right]$$

the integral is to be taken between $x = h$ and 0, for half an oscillation; and between these limits

$$\int_x \frac{x^n}{\sqrt{hx - x^2}} = -\,\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{3.1}{4.2}\cdot h^n\pi,$$

∴ time of an oscillation =

$$\pi\sqrt{\frac{a}{g}}\left\{1^2 + \left(\frac{1}{2}\right)^2\cdot\frac{h}{2a} + \left(\frac{1.3}{2.4}\right)^2\cdot\left(\frac{h}{2a}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2\cdot\left(\frac{h}{2a}\right)^3 + \&c.\right.$$

Cor. If the oscillations are very small, the time $= \pi\sqrt{\frac{a}{g}}$, as in the cycloid.

Now the length of the seconds pendulum (as will be shewn hereafter) can be determined by observation with great exactness, and hence the force of gravity is known, for it is equal to π^2. (length of seconds pendulum).

By this means g has been found $= 32.18$ feet.

114. Ex. 4. To find the time of oscillation in an hypocycloid, the force being in the centre of the globe, and varying as D.

Let S (Fig. 29) be the centre of the globe; EAD the base, A its middle point; B the lowest point of the hypocycloid; C the point from which the body begins to descend, CP the arc described in the time t; $\mu =$ absolute accelerating force, $s =$ BP, $v =$ vel. at P, $\beta =$ BC, $a =$ AS, $b =$ BS. Then the accelerating force down the curve $= \mu\,.\,\mathrm{PY} = \mu\,.\,\frac{a^2 s}{a^2 - b^2}$.

$$\therefore\ v d_s v = -\mu.\frac{a^2 s}{a^2 - b^2},$$

$$\therefore\ v^2 = \frac{a^2\mu}{a^2 - b^2}\,(\beta^2 - s^2),$$

$$\therefore\ t = \frac{\sqrt{a^2 - b^2}}{a\sqrt{\mu}}\cdot\int_s \frac{-1}{\sqrt{\beta^2 - s^2}} = \frac{\sqrt{a^2 - b^2}}{a\sqrt{\mu}}\cdot\cos^{-1}\frac{s}{\beta}.$$

$$\therefore\ \text{time of an oscillation} = \frac{\pi}{\sqrt{\mu}}\cdot\left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}}.$$

115. Ex. 5. If a particle acted on by gravity descend along the exterior, or being projected, ascend along the interior surface of a curve, situated in a vertical plane, it will fly off when the horizontal velocity is a maximum.

For if the horizontal ordinates of the curve increase faster than the horizontal motion of the body, the body will necessarily press against the curve, and the curve by its reaction will continually increase the horizontal velocity of the body; but these effects will cease as soon as the two motions become equal, and $\therefore$ the horizontal velocity will be a maximum, at the instant the particle ceases to press against the curve.

Ex. A body is projected along the quadrant of a circle with a given velocity; to find the point where it will leave the curve.

Case 1. Let it be projected from the vertex, along the exterior surface, with a velocity due to the altitude h; $a =$ the radius of the circle, $x =$ the distance of the body from the horizontal plane on which the quadrant stands.

$$\therefore \text{ velocity} = \sqrt{2g(h + a - x)} \propto \sqrt{h + a - x},$$

$$\therefore \text{ horizontal velocity} \propto x\sqrt{h + a - x},$$

which is a maximum when $x = \frac{2}{3} \cdot (a + h)$.

Case. 2. When it is projected from the horizontal plane, along the interior surface, the horizontal velocity $\propto x\sqrt{h - x}$, which will be a maximum when $x = \frac{2}{3} h$.

Cor. If a particle moving along a curve be acted on by any forces, it will fly off when the pressure or reaction $= 0$, (Art. 107).

116. Ex. 6. A particle, acted on by gravity, descends along a curve situated in a vertical plane; to find the form of the curve, that the pressure on it at every point may be the same.

Let the velocity at any point P (Fig. 30) in the curve be that due to its distance below the horizontal straight line Ax.

$x = \mathrm{AM}$, $y = \mathrm{MP}$, $s =$ arc described, $\mathrm{W} =$ weight of the particle. $\therefore$ (velocity)$^2 = 2gy$.

$$\therefore \text{pressure} = \mathrm{W}\, d_s x + \frac{2\,\mathrm{W}y}{rad.\ curv.} = \mathrm{W}\, d_s x + \frac{2\mathrm{W}y d_s^2 x}{d_s y};$$

which being constant, put it $= \mathrm{W}k$, and multiply by $\dfrac{d_s y}{2\sqrt{y}}$.

$$\therefore \frac{d_s x \,.\, d_s y}{2\sqrt{y}} + \sqrt{y} \,.\, d_s^2 x = \frac{k d_s y}{2\sqrt{y}}:$$

and integrating, $\sqrt{y} \,.\, d_s x = k\sqrt{y} + \mathrm{C} \,.\,.\,.\,.\,.\,.\,.\,.\, (1)$,

and by integrating this expression again, we shall have the equation of the curve in terms of x and y.

By dividing equation (1) by $\sqrt{y}$, and differentiating, we obtain the radius of curvature $= \dfrac{d_s y}{d_s^2 x} = -\dfrac{2y^{\frac{3}{2}}}{\mathrm{C}}$.

Case 1. Suppose the curve meets the axis of x at an $\angle\ \alpha$, $\therefore \mathrm{C} = 0$, and $d_s x = \cos\alpha$,

$$\therefore d_x y = \tan\alpha,$$
$$\therefore y = x\tan\alpha,$$

which is the equation of a straight line.

Case 2. When $k > 1$, C must be negative, as will appear from the expression for $d_x y$ found from equation (1).

Let $\therefore \mathrm{C} = -\sqrt{a}\,.(k^2 - 1)$, and $\sqrt{y} = v + k\sqrt{a}$; then, by means of equation (1), we shall have,

$$y = (v + k\sqrt{a})^2$$

and

$$\pm x.\left(1 - \frac{1}{k^2}\right)^{\frac{1}{2}} = \left(v + 2\frac{1+k^2}{k}.\sqrt{a}\right).\sqrt{a - v^2} + 3a\cos^{-1}\frac{v}{\sqrt{a}},$$

which two equations express the relation between x and y.

v is confined between the values $\pm\sqrt{a}$, and $\therefore$ the greatest and least values of y are $(k+1)^2 . a$, $(k-1)^2 . a$; and since $\pm d_x y = \dfrac{\sqrt{(k^2-1)(a-v^2)}}{\sqrt{a} + vk}$, it appears that when $v = \pm\sqrt{a}$, the curve is $\parallel$ to the axis of x, and as v increases from $-a$ to 0, the $\angle$ of inclination of the curve continually increases, and from 0 to a continually diminishes. This is the descending branch; and on account of the double sign with which $d_x y$ is affected, the ascending branch will be similar to it. It is evident, from the value of x, that there are an infinite number of similar parts, forming the curve in Fig. 30; the distance between corresponding points of two successive ascending, or descending branches, being

$$\frac{6ak\pi}{\sqrt{k^2-1}},$$

$$\because \cos\theta = \cos(2\pi+\theta) = \cos(4\pi+\theta) \ldots\ldots$$

The radius of curvature $= \dfrac{2y^{\frac{3}{2}}}{(k^2-1)\sqrt{a}}$.

Case 3. When $k = 1$, C is necessarily negative, put it $= -2\sqrt{a}$, and $v^2 = \sqrt{y} - \sqrt{a}$,

$$\therefore y = (v^2+\sqrt{a})^2.$$

$$\pm x = \frac{2v}{5\sqrt[4]{a}} \cdot (v^4 - 5a).$$

$$\text{and } \pm d_x y = \frac{2v\sqrt[4]{a}}{v^2-\sqrt{a}}.$$

Here when $v = 0$, $x = 0$, $y = a$, and the curve is || to the axis of x, and as v increases from 0 to $\sqrt[4]{a}$, x and y increase, and when $v = \sqrt[4]{a}$, the curve is || to the axis of y; and as v increases x diminishes, and y increases until $v = \infty$, when $x = \infty$, $y = \infty$, and the curve is || to the axis of x. The curve consists of two similar parts, similarly situated on contrary sides of the axis of y, as in Fig. 31.

The radius of curvature $= \dfrac{y^{\frac{3}{2}}}{\sqrt{a}}$.

Case 4. When k is < 1, C is either positive or negative.

Put $C = (1-k^2)\sqrt{a}$, and $\sqrt{y} - k\sqrt{a} = v$,

$$\therefore y = (v + k\sqrt{a})^2.$$

$$\pm x\left(\frac{1}{k^2}-1\right)^{\frac{1}{2}} = \left(v - 2.\frac{1+k^2}{k}.\sqrt{a}\right).\sqrt{v^2-a} + 3a.\log_e\left\{\frac{v}{\sqrt{a}} + \left(\frac{v^2}{a}-1\right)\right.$$

$$d_x y = \sqrt{1-k^2}.\frac{\sqrt{v^2-a}}{kv+\sqrt{a}}.$$

Hence the least value of v is $\sqrt{a}$, and $\therefore$ the least value of $y = (1 + k)^2 . \sqrt{a}$, and as v increases both y and x increase without limit, and the curve approximates to an $\angle$ of inclination to the axis of x, the cosine of which is k. Its form is that of Fig. 32.

$$\text{The radius of curvature} = \frac{-2y^{\frac{3}{2}}}{(1-k^2)\sqrt{a}}.$$

117. Def. If there be a system of curves described in one plane after a given law, the curve which cuts off from each, a part that would be described in the same time by a particle descending in them, by the action of a given force in that plane, is said to synchronize them.

Ex. 7. To find the curve which synchronizes all straight lines drawn from a given point, the descending particle being acted on by gravity.

Let AP (Fig. 33) be one of the lines, draw AB vertical, PM horizontal. $x = \text{AM}$, $y = \text{MP}$.

Then the force down AP $\propto \dfrac{\text{AM}}{\text{AP}}$ . . . . (Art. 58),

$$\text{and time down AP} \propto \frac{\text{AP}^2}{\text{AM}} \propto \frac{x^2 + y^2}{x} \text{ . . . . . (Art. 53),}$$

which being constant, put it $= 2a$,

$$\therefore x^2 + y^2 = 2ax,$$

$$\therefore y^2 = 2ax - x^2;$$

hence the curve required is a circle APB.

Ex. 8. To synchronize all cycloids having their cusps at a given point, and bases in the same horizontal line.

Let A be the given point (Fig. 34), AM the horizontal line; $\text{AM} = x$, $\text{MP} = y$, $a =$ the radius of the generating circle of the cycloid AP. Then time down AP $\propto \sqrt{a} . \text{vers}^{-1} \dfrac{y}{a}$; which being constant, put it equal to $\sqrt{k}$,

$$\therefore \frac{y}{a} = \text{vers} \sqrt{\frac{k}{a}},$$

also $\frac{x}{a} = \text{vers}^{-1}\frac{y}{a} - \sqrt{2.\frac{y}{a} - \frac{y^2}{a^2}} = \sqrt{\frac{k}{a}} - \sin\sqrt{\frac{k}{a}}$

From which two equations eliminating (a), we have the equation of the required curve.

Cor. In the cycloid $d_x y = \sqrt{\frac{2a}{y} - 1}$, and by differentiating the equation of the synchronous curve, we have $d_y x = -\sqrt{\frac{2a}{y} - 1}$; and therefore the curve BP cuts all the cycloids at right angles.

118. In cases where the time down the given curve cannot be found in finite terms, the following method may be applied.

Let $x = f(y, \alpha)$ be the equation of the curve, in which α is the parameter.

Then the time $\propto \int_y \frac{d_y s}{\sqrt{y}} \propto \int_y \frac{p}{\sqrt{y}}$, where $p\ (= d_y s)$ is deduced from the equation of the curve, and $\therefore$ contains α.

$$\therefore \int_y \frac{p}{\sqrt{y}} = 2k \ldots\ldots (1).$$

The integral of the first number of this equation is to be taken between y = constant, and y = MP in the required curve, and when taken between these limits, it is constant; and if we suppose α to become $\alpha + \delta\alpha$, then x becomes $x + \delta x$, and y, $y + \delta y$ ($\because$ x and y now belong not to the given but to the required curve). Hence, taking the variation of (1), and putting Q for $d_\alpha\left(\int_y \frac{p}{\sqrt{y}}\right)$, we have

$$\frac{p}{\sqrt{y}}.\,\delta y + Q\delta\alpha = 0 \ldots\ldots (2),$$

this equation being of the dimensions $(\frac{1}{2})$.

But if the parameter be so taken that the equation of the curve $x = f(y, \alpha)$ may be homogeneous with regard to α and y, equation (2) will also be homogeneous with regard to the same quantities, and ∴ by the property of homogeneous differential equations,

$$\frac{p}{\sqrt{y}} \cdot y + \mathrm{Q} . \alpha = k,$$

and eliminating Q between this and (2), dividing by $\delta\alpha$, and taking the limits,

$$\alpha \, d \, y - y + \frac{k\sqrt{y}}{p} = 0.$$

By eliminating (α) and (p), by means of $x = f(y, \alpha)$ we have the differential equation of the required curve.

119. **Def.** A *tautochronous* curve is one in which the time of an oscillation is the same whatever be the magnitude of the arc of oscillation.

Ex. 9. A particle is acted on by any forces in one plane; to find the tautochronous curve.

Resolve the forces into XY ‖ to the co-ordinate axes of xy. Then $(\text{vel.})^2 \propto \int (\mathrm{X}dx + \mathrm{Y}dy) \propto \phi(a, b) - \phi(x, y)$; (Art. 82),

$$\therefore \text{time} \propto \int \frac{ds}{\sqrt{\phi(a, b) - \phi(x, y)}},$$

$$\propto \int \frac{ds}{\sqrt{\phi(a, b)}} \left\{1 + \tfrac{1}{2} . \frac{\phi(x, y)}{\phi(a, b)} + \ldots . \right\}.$$

The general term of this $\propto \int \frac{\phi(x, y)^n \, ds}{\phi(a, b)^{n+\frac{1}{2}}}$, which taken between the limits $x, y = a, b$; and $x, y = a', b'$, (where a', b' are co-ordinates of the lowest point), ought to be independent of a, b. This will be the case, if

$$\int \frac{\phi(x, y)^n ds}{\phi(a, b)^{n+\frac{1}{2}}} \propto \left(\frac{\phi(x, y)}{\phi(a, b)}\right)^{n+\frac{1}{2}}.$$

or, differentiating both sides, and reducing

$$ds = \frac{d . \phi(x, y)}{\sqrt{\phi(x, y)}},$$

and by integrating,

$$s \propto \sqrt{\phi(x, y)} \propto \sqrt{\int (\mathrm{X}dx + \mathrm{Y}dy)},$$

$$\therefore\ sds \propto \mathrm{X}dx + \mathrm{Y}dy,$$

which is the general equation of the tautochronous curve.

Cor. If the force in direction of X be constant, and $\mathrm{Y} = 0$,

$$sds = gdx,$$

$$\therefore\ s^2 = 2gx + \mathrm{C},$$

which is the equation of the cycloid.

120. Ex. 10. AB (Fig. 35) is a given curve meeting the vertical AD at right $\angle^s$, at its lowest point A; to find the nature of the curve AC, so that the compound curve BAC may be tautochronous.

Let A be the origin, AD the axis of x; s, S, arcs of AB, AC to abscissa x; h = altitude from which the body descends.

$$\therefore\ \text{vel.} \propto \sqrt{h - x},$$

$$\therefore\ \text{time down BA} \propto \int \frac{ds}{\sqrt{h - x}}, \text{ between } x = h, \text{ and } x = 0;$$

$$\text{and time down CA} \propto \int \frac{d\mathrm{S}}{\sqrt{h - x}}, \text{ between the same limits;}$$

$$\therefore\ \text{time of an oscil.} \propto \int \frac{ds + d\mathrm{S}}{\sqrt{h - x}}, \text{ between the same limits;}$$

and by reasoning as in the last problem,

$$\int (ds + d\mathrm{S}) \frac{x^n}{h^{n+\frac{1}{2}}} \propto \left(\frac{x}{h}\right)^{n+\frac{1}{2}}.$$

$$\therefore\ ds + d\mathrm{S} \propto \frac{dx}{\sqrt{x}}$$

$$\therefore\ s + \mathrm{S} = \sqrt{ax}, \text{ is the equation required.}$$

For brachystochronous curves, see Airy's Mathematical Tracts, *Calculus of Variations*.

121. Ex. 11. A particle, acted on by any forces, is compelled to move along a given curve surface; to determine the motion.

From the equation of the given surface, suppose we deduce $d_xz = p$, $d_yz = q$. Let P = the reaction, which is ⊥ to the surface; v = the velocity.

$$\therefore m.d_t^2x = X - \frac{Pp}{\sqrt{1+p^2+q^2}};$$

$$m.d_t^2y = Y - \frac{Pq}{\sqrt{1+p^2+q^2}}; \quad m.d_t^2z = Z + \frac{P}{\sqrt{1+p^2+q^2}}.$$

Multiply respectively by $2dx$, $2dy$, $2dz$, and add, and integrate,

$$\therefore m.v^2 = d_tx^2 + d_ty^2 + d_tz^2 = C + 2\int(Xdx + Ydy + Zdz),$$

Again, multiply by $p, q, -1$, respectively, and add;

$$m.(pd_t^2x + qd_t^2y - d_t^2z) = Xp + Yq - z - P\sqrt{1+p^2+q^2}.$$

But $pd_tx + qd_ty - d_tz = 0$,

and therefore by differentiating we have,

$$pd_t^2x + qd_t^2y - d_t^2z = -d_tp.d_tx - d_tq.d_ty.$$

Consequently $P = \dfrac{Xp+Yq-Z}{\sqrt{1+p^2+q^2}} + m.\dfrac{d_tp.d_tx + d_tq.d_ty}{\sqrt{1+p^2+q^2}}.$

Eliminating P respectively between the first and third, and second and third of the original equations, we obtain

$$\left.\begin{aligned} m(d_t^2x + pd_t^2z) &= X + Zp \\ m(d_t^2y + qd_t^2z) &= Y + Zq \end{aligned}\right\} \ldots (A),$$

and by eliminating t between these equations, the resulting equation, combined with that of the given surface, will give the path of the particle.

Cor. 1. If the particle be acted on by no forces, the equations (A) become

$$d_t^2x + pd_t^2z = 0, \quad d_t^2y + qd_t^2z = 0;$$

changing the independent variable and eliminating t, these give

$$\frac{p + d_zx}{q + d_zy} = \frac{d_z^2x}{d_z^2y}.$$

Cor. 2. The equation $m \,.\, v^2 = \mathrm{C} + \int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$ is exactly the same as if the body had been free (Art. 82), and the same conclusions follow from it; viz.—

$1.^{st}$ That, when the particle is acted on by no forces, the *vis viva* is constant.

$2.^{nd}$ That, when the forces are functions of the co-ordinates only, the *vis viva* acquired is independent of the surface on which the particle moves.

$3.^{rd}$ That the path is such, that $\int(mvds)$ or $\int_t(mv^2)$ is a minimum.

122. Ex. 12. A particle, acted on by no forces, moves on a curve of double curvature; to find the pressure.

Let c = the velocity, which is uniform (last Cor.), m = the mass of the particle.

$$\therefore\ c^2 = d_t x^2 + d_t y^2 + d_t z^2,$$

$$\text{and, } d_t^{\,2}x = d_t s^2 \,.\, d_s^{\,2}x = c^2 d_s^{\,2}x.$$

$$\text{Similarly, } d_t^{\,2}y = c^2 d_s^{\,2}y, \text{ and } d_t^{\,2}z = c^2 d_s^{\,2}z,$$

$$= m\sqrt{(d_t^{\,2}x)^2 + (d_t^{\,2}y)^2 + (d_t^{\,2}z)^2},$$

$$\therefore \text{ the pressure} = mc^2\sqrt{(d_s^{\,2}x)^2 + (d_s^{\,2}y)^2 + (d_s^{\,2}z)^2},$$

$= m . \dfrac{(\text{velocity})^2}{\text{rad. curvature}}$ = the pressure due to the centrifugal force. (Art. 108). Hymer's Anal. Geom. art. 66. cor. 1.

Cor. 1. If the particle be acted on by any forces, the whole pressure on the curve will (Art. 17) be

= pressure due to the impressed forces,
+ pressure due to the centrifugal force.

Cor. 2. If the particle move on a surface, acted on by any forces, the pressure will be equal to that which it exerts on the curve of double curvature which it describes, resolved in a direction perpendicular to the surface. The analytical expression for this pressure is given in last Article.

COR. 3. When the particle moves on a surface acted on by no forces, the velocity is uniform; and since the reaction is ⊥ to the surface, the centrifugal force must also be ⊥ to the surface; ∴ the plane which contains the osculating circle of the curve of double curvature described on the surface, is always ⊥ to the given surface, and ∴ the curve is the shortest that can be described; which result may be derived at once from the principle of least action. (Cor. 2, Art. 121).

123. Ex. 13. As an interesting application of the general principle of Art. 107, we may consider the motion of a particle in the interior of a spherical bowl, when acted on by gravity only.

Taking the centre for the origin, the equation of the surface is

$$x^2 + y^2 + z^2 = a^2 \quad \ldots\ldots \quad (1),$$

$$\therefore\ p = -\frac{x}{z},\ q = -\frac{y}{z}, \text{and } \sqrt{1 + p^2 + q^2} = \frac{a}{z}, \quad \text{and } \therefore$$

the equations of motion are,

$$d_t^2 x = \frac{P}{m} \cdot \frac{x}{a},\quad d_t^2 y = \frac{P}{m} \cdot \frac{y}{a},\quad d_t^2 z = \frac{P}{m} \cdot \frac{z}{a} + g.$$

Multiplying respectively by $2dx$, $2dy$, $2dz$, and integrating, we have

$$(\text{vel.})^2 = d_t x^2 + d_t y^2 + d_t z^2 = 2gz + C \quad \ldots\ldots \quad (2).$$

Again, multiply by x, y, z respectively, and add;

$$\therefore\ x d_t^2 x + y d_t^2 y + z d_t^2 z = \frac{P}{m} \cdot a + gz.$$

But the equation of the sphere, being twice differentiated, gives

$$x d_t^2 x + y d_t^2 y + z d_t^2 z = -d_t x^2 - d_t y^2 - d_t z^2 = -2gz - C,$$

$$\therefore\ \frac{P}{m} = -\frac{3gz + C}{a} \quad \ldots\ldots\ldots \quad (3).$$

Again, multiply the first by y, the second by x, subtract and integrate,

$$\therefore\ y d_t x - x d_t y = h.$$

Now the equation of the surface gives $x d_t x + y d_t y = -z d_t z$;

CHAPTER VI.

ON THE FREE MOTION OF SEVERAL UNCONNECTED PARTICLES.

The principle of the conservation of the motion of the centre of gravity.

125. If any forces whatever act upon the unconnected particles of a system, their common centre of gravity will move in the same manner as a body equal to the sum of all the particles acted on by the same forces.

Let $\dot{x}\dot{y}\dot{z}$ be the co-ordinates of the common centre of gravity; xyz those of any particle m, referred to the centre of gravity as origin and co-ordinates ∥ to the former. $X\,Y\,Z$ the resolved parts of the moving forces acting on m ∥ to the co-ordinates.

Hence the co-ordinates of m are $\dot{x} + x$, $\dot{y} + y$, $\dot{z} + z$, and $\therefore\ m\,(d_t^2\dot{x} + d_t^2 x) = X$, $\therefore\ d_t^2\dot{x}\,.\,\Sigma m + \Sigma\,(m d_t^2 x) = \Sigma X$.

But $\Sigma\,(mx) = 0$, and $\therefore\ \Sigma\,(m d_t^2 x) = 0$;

$$\therefore\ d_t^2\dot{x} = \frac{\Sigma X}{\Sigma m}; \quad \text{similarly}, d_t^2\dot{y} = \frac{\Sigma Y}{\Sigma m},\ d_t^2\dot{z} = \frac{\Sigma Z}{\Sigma m};$$

which equations are the same as for the motion of the mass Σm, acted on by all the forces which act on the particles.

126. Cor. 1. If the particles are acted on by no other forces than their mutual influences, the centre of gravity will either remain at rest or move uniformly forward in a straight line. For the body acting, and that acted upon, exert equal forces

on each other in opposite directions (Art. 24), and ∴ the sum of the forces acting on the particles must be $= 0$, and by the Law of Inertia, a body, acted on by no forces, will either remain at rest, or move uniformly in a straight line.

127. Cor. 2. A system of particles cannot, by their action upon each other, alter the motion of their common centre of gravity.

This principle is true, if the particles impinge against each other during their motions.

128. If the particles of a system be acted on by no other forces than their mutual attractions, the expression $\Sigma\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$ will be a complete differential.

For let m, m' be two of the particles, r their distance from each other; xyz, $x'y'z'$ their co-ordinates; $m'f$ the absolute accelerating force of m'. Then

$$r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2,$$

and the accelerating force of m' on m, in direction of $x =$

$$\frac{\mathrm{X}}{m} = m'f.\frac{x' - x}{r} = m'fd_xr,$$

$\therefore \mathrm{X} = mm'fd_xr$; similarly, $\mathrm{Y} = mm'fd_yr$, $\mathrm{Z} = mm'fd_zr$.

$\therefore \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = mm'f(d_xr\,.\,dx + d_yr\,.\,dy + d_zr\,.\,dz)$;

and, in like manner,

$\mathrm{X}'dx' + \mathrm{Y}'dy' + \mathrm{Z}'dz' = mm'f(d_{x'}r\,.\,dx' + d_{y'}r\,.\,dy' + d_{z'}r\,.\,dz')$,

$\therefore \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz + \mathrm{X}'dx' + \mathrm{Y}'dy' + \mathrm{Z}'dz' = mm'fdr.$

where the quantities $\mathrm{X}'\mathrm{Y}'\mathrm{Z}'$ refer to the action of m on m'.

Hence, $\Sigma\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) = \Sigma\,(mm'fdr)$.

But f being a function of r, the right hand side of the equation is a complete differential, and ∴ the expression $\Sigma\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$ is a complete differential.

129. Cor. The *vis viva* $= \Sigma\,(mv^2) = \mathrm{C} + 2\Sigma\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$; and since the last expression is a function of

$x_0 y_0 z_0$; $x'y'z'$, &c. . . . it follows, from similar reasoning to that employed in Art. 82, that the *vis viva* acquired by the system in passing from any one given position to another, is altogether independent of the paths taken by its particles, providing none of them impinge against each other (Art. 33); that the *vis viva* of the system is always the same whenever it comes to the same position; and that the paths of the particles will be such that $\Sigma \int (mv\,ds)$ or $\Sigma \int_t (mv^2)$ is less than if they had taken any other courses.

The Principle of the Conservation of Areas.

130. If there be a system of free particles, influenced only by their mutual actions, the sum of the products of each into the projection (on a given plane) of the area described by its radius vector round a fixed point, is proportional to the time.

Let xyz be the co-ordinates of the particle m; A_x, A_y, A_z, the projections of the area described by it round the origin upon the planes of yz, xz, xy. Then $\because$ by the mutual actions of any two particles, equal moving forces are generated in opposite directions, $\therefore$ the sum of the moments of the impressed moving forces about any axis will be equal to 0.

Now $\Sigma\,(md_t^2x) = \Sigma X$, $\Sigma\,(md_t^2y) = \Sigma Y$, $\Sigma\,(md_t^2z) = \Sigma Z$;

and since $\Sigma\,(Yz - Zy) = 0$,

$$\therefore \Sigma m\,(zd_t^2y - yd_t^2z) = 0,$$

and integrating $\Sigma m\,(zd_ty - yd_tz) = C_1$ or $\Sigma\,(md_tA_x) = C_1$

and integrating again,

$$\Sigma\,(m\,.\,A_x) = C_1t;\ \text{similarly, } \Sigma\,(m\,.\,A_y) = C_2t,\ \Sigma\,(m\,.\,A_z) = C_3t.$$

131. Cor. 1. The same property is true, if the origin instead of being a *fixed* point be in the centre of gravity of the system.

132. Cor. 2. If a plane be drawn making with the axes of xyz angles whose cosines are $\frac{\Sigma m A_x}{\Sigma m A}$, $\frac{\Sigma m A_y}{\Sigma m A}$, $\frac{\Sigma m A_z}{\Sigma m A}$, where $(\Sigma m A)^2 = (\Sigma m A_x)^2 + (\Sigma m A_y)^2 + (\Sigma m A_z)^2$; this plane will retain the same position during all the motions which the particles undergo; it is called the *invariable plane,* and sometimes the *principal plane of moments.*

133. Cor. 3. A is the projection of the area described by m on the invariable plane. (Hymer's Anal. Geom. art. 18).

134. Cor. 4. The sum of the products of each particle into the projection of the area described by it, is greater for the invariable plane than for any other, and the sum on a plane $\perp$ to it $= 0$, and on a plane inclined at an $\angle\ \theta$ to it $=$ $(\Sigma m A) \cos \theta$. (Hymer's Anal. Geom. art. 18).

To this Chapter properly belong the problems of two and three bodies, but as they form a distinct treatise of themselves, they are omitted, and the reader is referred to Airy's Mathem. Tracts, *Lunar Theory.*

G the centre of gravity of the body; join mG and produce it to n, making Gn = Gm; and at n in opposite directions ‖ to mP apply the forces 3, 4, each equal to P; these cannot alter the motion. Now the forces (1, 4) can produce no effect on the motion of the centre of gravity, for being similarly situated with regard to it, and acting in opposite directions, if one move G forward, the other will move it as far backwards; but the remaining forces (2, 3), by STATICS, produce the same effect on the body as a single force = 2P, applied at G. Again, (2, 3) can produce no rotation about G, for if one turn the body one way, the other will turn it as much the other way, ∴ (2, 3) cause no rotation, but (1, 4) will evidently tend equally to turn the body round G, and ∴ the rotation about G will be the same as if 2P acted at m, and G were fixed.

The Principle of the Conservation of Vis Viva.

138. Let m be one of the particles of a system of bodies. Then if to the forces which are impressed upon m we add those which it suffers from the action of the other parts of the system, we may consider it as free. Let X Y Z be the moving forces, so estimated, acting upon it, resolved in directions of the co-ordinates. Then

$$m \,.\, d_t^2 x = \mathrm{X}, \;\; m \,.\, d_t^2 y = \mathrm{Y}, \;\; m \,.\, d_t^2 z = \mathrm{Z}.$$

$$\therefore m\,(d_t x^2 + d_t y^2 + d_t z^2) = \mathrm{C} + 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz);$$

or, putting v for the velocity of m,

$$\Sigma\,(mv^2) = \mathrm{C} + 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz).$$

Now if $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ be an exact differential of a function of three variables, we have

$$\Sigma\,(mv^2) = \mathrm{C} + 2\,\Sigma\phi\,(x, y, z).$$

This equation is similar to the one we have already found in Art. 82, and leads to similar results:—That if the system be not acted on by any forces, its *vis viva* is constant; that, in other cases, the increment or decrement of its *vis viva*, in passing from one position to another, is independent of the paths described by its different particles, and is therefore the same as if the particles were free; that this increment or decrement is evanescent every time the system returns to the same position.

It appears, from the preceding Chapters, that the same is true if the particles be unconnected and attract each other, or be attracted to a fixed centre, or to the centre of gravity of the system, which comprehend all the forces which are found in Nature. See Arts. 59, 63, 82, 111, 129.

139. The principle of *vis viva* is not true if any of the bodies move in a resisting medium, or be affected by friction; for then $\Sigma\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$ is not a complete differential. And of this we may also be assured, by considering that either of these causes tends continually to diminish the velocity, and ∴ the *vis viva*; and, supposing the system acted on by no other forces, would at length reduce it to a state of rest; whereas, according to the general principle, the *vis viva* ought to be constant.

If any of the bodies happen to impinge upon each other, the *vis viva* (unless they are perfectly elastic) will be suddenly diminished. (Art. 33).

140. If a body (or system of rigidly connected particles) move in any manner whatever, its *vis viva* at any instant = the *vis viva* of the whole mass collected at the centre of gravity + the *vis viva* round the centre of gravity; or, in other words, the *vis viva* of a body = the *vis viva* from translation + the *vis viva* from rotation.

For let $\dot{x}\,\dot{y}\,\dot{z}$ be co-ordinates of the centre of gravity of the system, the axis of y being ∥ to the axis of rotation round the centre of gravity; r = distance of any particle m from that axis; $\theta = \angle$ which r makes with plane xy.

$$\therefore\ x = \dot{x} + r\cos\theta,\ z = \dot{z} + r\sin\theta, \text{ and } d_ty = d_t\dot{y},$$

and (vel.)^2 of the particle =
$d_t x^2+d_t y^2+d_t z^2=(d_t\dot{x}-r\sin\theta.d_t\theta)^2+d_t\dot{y}^2+(d_t\dot{z}+r\cos\theta.d_t\theta)^2$
$=d_t\dot{x}^2+d_t\dot{y}^2+d_t\dot{z}^2+r^2d_t\theta^2-2d_t\dot{x}.d_t\theta.r\sin\theta+2d_t\dot{z}.d_t\theta.r\cos\theta.$

Now $\Sigma(m.r\sin\theta)=0$, $\Sigma(m.r\cos\theta)=0$, and

∴ the *vis viva* $=\Sigma m\,(d_t\dot{x}^2+d_t\dot{y}^2+d_t\dot{z}^2)+\Sigma m\,(rd_t\theta)^2$,

= the *vis viva* from translation + the *vis viva* from rotation.

Cor. The *vis viva* from rotation $=\Sigma mr^2d_t\theta^2=d_t\theta^2\Sigma mr^2$ = the moment of inertia multiplied by the square of the angular velocity.

141. When a system passes through a position of equilibrium its *vis viva* will be a maximum or minimum.

By D'Alembert's principle, the effective forces are always such as would, when applied in opposite directions to those in which they act, balance the impressed forces; and here the latter would produce equilibrium, and ∴ the former are such as would also, in the same position of the system, produce equilibrium.

Let v = the velocity of one of the particles m; ∴ the effective force on m is md_tv, and since there is an equilibrium among the effective forces, ∴ by virtual velocities,* $\Sigma(mvd_tv)=0$, ∴ $\Sigma(mv^2)$ is a maximum or minimum.

142. Cor. If the position be one of *stable* equilibrium, when the system passes out of that position, the tendency of the impressed forces will be to bring it back to it again; ∴ they will diminish both the $\angle^r$ velocity (for by $\angle^r$ motion the body is carried from that position) and the velocity of translation; and ∴ the *vis viva*, which = *vis viva* from rotation

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* For a demonstration of the Statical principle of virtual velocities, the reader is referred to Pritchard's Statical Couples, *Appendix*, *Prop. D.*

+ that from translation, will be diminished. And, by similar reasoning, if the position be one of *unstable* equilibrium, the *vis viva* will be increased; but if of *neuter* equilibrium, it will neither be increased nor diminished. Hence the *vis viva* will be a *maximum*, *minimum*, or *neither*, according as the position is one of *stable*, *unstable*, or *neuter* equilibrium.

The Principle of Least Action.

143. This principle which has been proved to be true for the motion of a single particle, is also true for any system of particles.

For, as in Art. 84, we have

$$\delta(v\,ds) = d_t(d_t x\,\delta x + d_t y\,\delta y + d_t z\,\delta z);$$

$$\therefore\ \delta\int(v\,ds) = d_t x\,\delta x + d_t y\,\delta y + d_t z\,\delta z + \text{constant},$$

$$\therefore\ \delta\,.\,\Sigma\int(mv\,ds) = \Sigma m\,(dx\,\delta x + dy\,\delta y + dz\,\delta z) + \mathrm{C};$$

$$\text{or, } \delta\,.\,\Sigma\int_t(mv^2) = \Sigma m\,(dx\,\delta x + dy\,\delta y + dz\,\delta z) + \mathrm{C}.$$

The extreme points of the path being fixed, δx, δy, δz at these limits $= 0$, and $\therefore\ \delta\,.\,\Sigma\int(mvds)$ or $\delta\,\Sigma\int_t(mv^2) = 0$, and $\therefore\ \Sigma\int(mvds)$ or $\Sigma\int_t(mv^2)$ is a minimum. This is the principle of least action, and may be stated in words thus;—*The sum of the vires vivæ, during the time that a system employs in passing from one given position to another, is less than if the particles had taken any other courses.*

This principle depends upon $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ being a complete differential of a function of three variables, and $\therefore$ does not hold when any of the particles are subject to friction, or move in a resisting medium.

144. Def. If a system of particles $m, m_1, m_2 \ldots$ revolve round an axis, and $r, r', r'', \ldots$ be their distances from that axis, $\Sigma\,(mr^2)$ is called the moment of inertia of the system.

A system of rigidly connected particles, moveable about a fixed axis, is acted on by forces in parallel planes perpendicular to the axis; to determine the angular motion.

Let F be the force acting upon the particle m, p the $\perp^r$ from the axis upon its direction, θ the $\angle$ through which the body has revolved in the time t.

The actual velocity of m is $rd_t\theta$, $\therefore$ the *effective* moving force on $m = mrd_t^2\theta$ (Art. 77), and by D'Alembert's principle, the effective and impressed forces are such as would balance each other, $\therefore$ by STATICS

$$\Sigma(\mathrm{F}p) = \Sigma(mrd_t^2\theta \,.\, r) = d_t^2\theta \,.\, \Sigma(mr^2).$$

$$\therefore d_t^2\theta = \frac{\Sigma(\mathrm{F}p)}{\Sigma(mr^2)} = \frac{\text{moment of the forces}}{\text{moment of inertia}}.$$

145. COR. If a particle m' be added to the system at the distance r' from the axis, the numerator remains the same, but the term $(m'r'^2)$ is added to the denominator, $\therefore$ m''s effect on the system is measured by $m'r'^2$, which is $\therefore$ called its moment of inertia.

146. Required the quantity of matter which placed at the distance b from the axis shall resist the rotation as much as m placed at the distance a.

Let $x =$ the mass required; then the effects of x and m on the system are measured by xb^2 and ma^2; which being equal, we have

$$x = m \,.\, \frac{a^2}{b^2}.$$

147. To find the distance k (called the *radius of gyration*) from the axis at which the whole mass must be collected, so that the resistance to rotation may be the same as before.

The resistance of the whole mass placed at distance k is measured by $k^2 \,.\, \Sigma m$, which must $= \Sigma(mr^2)$.

$$\therefore k^2 = \frac{\Sigma(mr^2)}{\Sigma m} = \frac{\text{moment of inertia}}{\text{mass}}.$$

148. To find the centre of oscillation of a given rigid system.

Def. The *centre of oscillation* is that point in a body oscillating about a fixed axis, at which if the whole mass were collected, the time of oscillation would remain unaltered.

Let the plane of the paper pass through the centre of gravity G, and cut the axis of suspension perpendicularly in S (Fig. 39), and a vertical plane through the axis in SN; O the centre of oscillation, which must be in a plane passing through G and the axis, $\because$ when SG coincides with SN the body is acted on by no angular accelerating force; m any particle of the body, $mN \perp SN$; $x = SN$, $y = mN$, $r = Sm$, $\theta = mSN$, and $t =$ time from a given epoch.

$\therefore$ effective force on $m = mr d_t^2\theta$ . . . (Art. 144),
and impressed force on $m = mg$.

But, by D'Alembert's principle, the impressed would balance the effective forces,

$$\therefore \Sigma(mr^2 d_t^2\theta) = \Sigma(mgy) = g \,.\, \text{SG} \,.\, \sin\theta \,.\, \Sigma m,$$

$$\therefore d_t^2\theta = \frac{\text{SG} \,.\, \sin\theta \,.\, \Sigma m}{\Sigma(mr^2)} \,.\, g.$$

But when the whole mass is collected at O, we have, from the same formula,

$$d_t^2\theta = \frac{\text{SO} \,.\, \sin\theta \,.\, \Sigma m}{\text{SO}^2 \,.\, \Sigma m} \,.\, g,$$

$$\therefore \text{SO} = \frac{\Sigma(mr^2)}{\text{SG} \,.\, \Sigma m} = \frac{\text{moment of inertia}}{\text{SG} \,.\, \text{mass}}.$$

The point O is not necessarily in the plane of the paper, but its locus is a straight line || to the axis of suspension.

149. The centres of oscillation and suspension are convertible; that is, if the line of oscillation be made the axis of suspension, the former axis of suspension becomes the line of oscillation.

In Art. 154 it will be shewn, that the moment of inertia round S = mom. inert. round the centre of gravity.

+ mom. inert. of the mass collected at the centre of gravity.

$\therefore\ \Sigma(mr^2) = k^2 . \Sigma m + SG^2 . \Sigma m$, where k is the radius of gyration.

$$\therefore\ SO = \frac{\Sigma m . (k^2 + SG^2)}{SG . \Sigma m} = \frac{k^2}{SG} + SG \ldots \text{(Art. 148)},$$

$$\therefore\ GO . SG = k^2,\ \therefore\ SG = \frac{k^2}{GO},$$

and

$$OS = \frac{k^2 + GO^2}{GO} = \frac{\Sigma m . (k^2 + GO^2)}{GO . \Sigma m} = \frac{\text{mom. inert. round O}}{GO . \text{mass}}.$$

$\therefore$ when O is the centre of suspension, S is the centre of oscillation.

150. To find about what axis of suspension, the time of oscillation is the least possible.

The time will be the least possible when SO, or $SG + \frac{k^2}{SG}$, is a minimum.

$$\therefore\ SG = k.$$

Hence the locus of S is a cylinder, whose axis is ∥ to the axis of suspension, and base a circle, in the plane of oscillation, whose centre is G and radius k.

151. The property of pendulums, proved in Art. 149, has been ingeniously employed by Captain Kater to determine the length of the seconds pendulum. The following is a general outline of the method employed.

Let G (Fig. 40) be the centre of gravity of a pendulum, furnished with two ∥ cylindrical axes A, B, of equal radii, about either of which it can be made to oscillate. A large sliding weight C is attached to the pendulum, by means of which the times of oscillation about the two axes are made nearly equal; besides which there is a very small sliding weight m, which is placed in such a position, that when moved, the times of oscillation about the two axes are either both increased or both diminished (Art. 152); but more so about one than the other; this position is chosen with the view of increasing the play of the weight m, and thereby securing greater accuracy than if it were in such a position as to increase the time about one axis and diminish it about the other. The oscillations are very small, and the axis rests on two perfectly horizontal planes, and does not slide but roll; and, in the result, allowance is made for the buoyancy of the air; its resistance is neglected, for as will be shewn hereafter, it produces no sensible effect upon the time of an oscillation.

The number of oscillations about A in a day is observed, and also about B; and by means of the small weight m these numbers are made equal; the least distance between the cylindrical surfaces of the axes is the length of the simple pendulum. (Art. 153). It is, however, more convenient in practice for the pendulum to rest on knife edges instead of cylindrical axes; and instead of using the small sliding weight m, to equalize the number of oscillations about the axes by filing away small portions from the ends of the pendulum.

Let n = number of oscillations about either axis in the time t, l = the distance between the axes; ∴ the time of an oscillation $= \frac{t}{n}$.

Now (Art. 112) $\frac{t^2}{n^2} : 1^2 :: l : \frac{n^2 l}{t^2}$ = the length of the seconds pendulum.

Cor. The same experiment will enable us to determine the force of gravity; for $\frac{t}{n} = \pi \sqrt{\frac{l}{g}}$, and

$$\therefore g = \frac{n^2 \pi^2 l}{t^2}.$$

152. To determine the position of the small weight (m) in last article, that the times about the two axes may be either both increased or both diminished by moving it.

radius of gyration about G; then we may for an instant suppose the body to oscillate about E.

$$\therefore d_t^2\theta = \frac{\text{moment of the forces}}{\text{moment of inertia}} \text{ (Art. 144)},$$

$$= \frac{\text{M} . a\text{G} . \sin\theta}{\text{M}(k^2 + \text{EG}^2)} = \frac{a\text{G} . \theta}{k^2 + \text{AG}^2}, \text{ very nearly},$$

$\therefore$ the length of the simple pendulum $(= x) = \frac{k^2 + \text{AG}^2}{a\text{G}}$,

as will appear by integration;

$$\therefore x . a\text{G} = k^2 + \text{AG}^2.$$

Similarly, when the pendulum oscillates about the other axis,

$$x . b\text{G} = k^2 + \text{BG}^2;$$

and by subtracting this equation from the former, and observing that $a\text{G} - b\text{G} = \text{AG} - \text{BG}$, we have

$$x = \frac{\text{AG}^2 - \text{BG}^2}{\text{AG} - \text{BG}} = \text{AG} + \text{BG} = \text{AB}.$$

Let $l' =$ the length of the simple pendulum, when A is the axis of suspension, and the small weight at m; $\mathrm{M} =$ mass of the pendulum without (m), $k =$ the radius of gyration.

$$\therefore l' = \frac{\mathrm{M}(k^2 + \mathrm{AG}^2) + m.(\mathrm{A}m)^2}{\mathrm{M}.\mathrm{AG} + m.\mathrm{A}m} \ldots\ldots (\text{Art. } 148).$$

and if $l + \delta l'$ be the length of the simple pendulum, whem m is moved through the very small space $\delta.(\mathrm{A}m)$, we have, by taking the variation of the last expression,

$$\delta l' = -\mathrm{M}m.\frac{k^2 + \mathrm{AG}.\mathrm{G}m}{(\mathrm{M}.\mathrm{AG} + m.\mathrm{A}m)^2}.\delta(\mathrm{A}m).$$

Hence the time of an oscillation is always diminished about that axis, between which and the centre of gravity the small weight m is placed, by moving the weight from the axis, and increased by moving it towards it.

Let l'' be the length of the simple pendulum, when B is the axis, and the weight is at m. Then, by writing BG for AG, and $\delta(\mathrm{B}m)$ for $\delta(\mathrm{A}m)$ in the last expression, we have

$$\delta l'' = -\mathrm{M}m.\frac{k^2 - \mathrm{BG}.\mathrm{G}m}{(\mathrm{M}.\mathrm{BG} + m.\mathrm{B}m)^2}.\delta(\mathrm{B}m);$$

or, since $\delta(\mathrm{B}m) = -\delta(\mathrm{A}m)$,

$$\delta l'' = -\mathrm{M}m.\frac{\mathrm{BG}.\mathrm{G}m - k^2}{(\mathrm{M}.\mathrm{BG} + m.\mathrm{B}m)^2}.\delta(\mathrm{A}m).$$

Hence the time will be increased or diminished about both axes, if $\mathrm{BG}.\mathrm{G}m > k^2$, $\therefore m$ is any point between A and G, such that

$$\mathrm{G}m > \frac{k^2}{\mathrm{BG}}.$$

153. To shew that the length of the simple pendulum is equal to the distance between the surfaces of the axes of suspension.

Let a, b (Fig. 41) be the centres of the axes of suspension; aA, bB their radii; CE the horizontal plane on which the axis rests, aE vertical, G the centre of gravity of the *whole* pendulum, $\mathrm{M} =$ its mass; $\theta = \angle \mathrm{A}a\mathrm{E}$ which is very small, $k =$ the radius of gyration about G; then we may for an instant suppose the body to oscillate about E.

$$\therefore d_t^2\theta = \frac{\text{moment of the forces}}{\text{moment of inertia}} \text{ (Art. 144)},$$

$$= \frac{\text{M}\,.\,a\text{G}\,.\,\sin\theta}{\text{M}\,(k^2 + \text{EG}^2)} = \frac{a\text{G}\,.\,\theta}{k^2 + \text{AG}^2}, \text{ very nearly},$$

$$\therefore \text{ the length of the simple pendulum } (= x) = \frac{k^2 + \text{AG}^2}{a\text{G}},$$

as will appear by integration;

$$\therefore x\,.\,a\text{G} = k^2 + \text{AG}^2.$$

Similarly, when the pendulum oscillates about the other axis,

$$x\,.\,b\text{G} = k^2 + \text{BG}^2;$$

and by subtracting this equation from the former, and observing that $a\text{G} - b\text{G} = \text{AG} - \text{BG}$, we have

$$x = \frac{\text{AG}^2 - \text{BG}^2}{\text{AG} - \text{BG}} = \text{AG} + \text{BG} = \text{AB}.$$

CHAPTER VIII.

ON FINDING THE MOMENT OF INERTIA OF DIFFERENT BODIES.

154. The moment of inertia round any axis, is equal to the moment round an axis || to it through the centre of gravity, together with the moment of the whole mass collected at the centre of gravity.

For let the axis of rotation be that of $\dot{z}$; $\dot{x}\dot{y}\dot{z}$ being co-ordinates of the centre of gravity; $\dot{r}$ its distance from the axis; xyz co-ordinates of any particle m, reckoning from the centre of gravity as origin, r its distance from the axis; and r' its distance from the axis of z.

Then $\dot{r}^2 = \dot{x}^2 + \dot{y}^2$, $r'^2 = x^2 + y^2$, and $r^2 = (\dot{x} + x)^2 + (\dot{y} + y)^2$,

$$\begin{aligned} \therefore \Sigma(mr^2) &= \Sigma(mr'^2) + \Sigma(m\dot{r}^2) + 2\dot{x}\,\Sigma(mx) + 2\dot{y}\,\Sigma(my) \\ &= \Sigma(mr'^2) + \Sigma(m\dot{r}^2), \end{aligned}$$

because $\Sigma mx = 0$, and $\Sigma my = 0$.

155. Cᴏʀ. The moment of inertia about the centre of gravity is less than about any other axis whatever; that is, $\Sigma(m\dot{r}^2)$ is a minimum.

156. The following are examples of finding the moment of inertia of different bodies, round certain axes, and Art. 154 will thence enable us to determine the moments round any other axes parallel to them.

Ex. 1. To find the moment of inertia of a uniform straight line about one end.

Let a = its length, M = its mass; x = the distance of an element δx from the axis; the mass of that element = $\mathrm{M}\cdot\frac{\delta x}{a}$,

$$\therefore \text{ its moment of inertia} = \mathrm{M}.\frac{x^2\delta x}{a}.$$

$$\therefore \text{the moment of inertia of the line} = \mathrm{M}.\int_{x=a}\frac{x^2}{a} = \mathrm{M}.\frac{a^2}{3}.$$

Cor. The distance of the centre of oscillation from the axis, or the length of the simple pendulum $= \frac{\mathrm{M}.\frac{1}{3}a^2}{\mathrm{M}.\frac{1}{2}a} = \frac{2}{3}.a.$

Ex. 2. To find the moment of inertia of a circular arc about a diameter that bisects it.

Let C (Fig. 42) be the centre, BAD the arc, and AC the diameter that bisects it; A the origin of co-ordinates, AC the axis of x; a = AC, δs = an element of the arc; BAD = b, M = its mass.

$$\therefore \text{ mass of element } \delta s = \mathrm{M}.\frac{\delta s}{b}.$$

$$\text{and } \therefore \text{ the moment of this element round AC} = \mathrm{M}.\frac{y^2\delta s}{b}.$$

$$\therefore \text{moment} = \mathrm{M}.\int_x\frac{y^2 d_x s}{b} = \mathrm{M}.\frac{a}{b}.\int_x y = \mathrm{M}\cdot\frac{a}{b}\cdot\text{area BAD},$$

$$= \frac{\mathrm{M}}{2}\cdot\frac{a}{b}\cdot(ab - a.\mathrm{BD}),$$

$$= \mathrm{M}.\frac{a^2}{2}\left(1 - \frac{\text{chord}}{\text{arc}}\right).$$

Ex. 3. To find the moment of inertia of the arc BAD about its middle point A.

$\mathrm{M}a^2$ = moment round C = moment round G + $\mathrm{M}.\mathrm{GC}^2$. (Art. 154).

and mom. round A $= \text{mom. round G} + \text{M}.\text{AG}^2$,

$$= \text{M}a^2 - \text{M}\,(\text{GC}^2 - \text{AG}^2),$$
$$= \text{M}a^2 - \text{M}a\,(a - 2.\text{AG}) = \text{M}.2a.\text{AG},$$
$$= \text{M}.2a^2.\left(1 - \frac{\text{chord}}{\text{arc}}\right).$$

COR. The length of the simple pendulum $= \dfrac{\text{M}.2a.\text{AG}}{\text{M}.\text{AG}}$ $= 2a$, and $\therefore$ the time of oscillation will be the same, whatever be the length of the arc.

Ex. 4. To find the moment of inertia of a plane triangle about an axis, passing through one of its angular points, perpendicular to its plane.

Let A (Fig. 43) be the angle; draw Ay $\parallel$ CB for the axis of y, and take AC for the axis of x; $\text{M} =$ the mass of the $\triangle$, $\text{A} =$ its area, and xy co-ordinates of any point in it, $\theta =$ the $\angle$ ACB.

$$\therefore \text{ element of the mass} = \text{M}.\frac{\delta x\,\delta y}{\text{A}}.\sin\theta,$$

$$\therefore \text{ elem. of mom. round A} = \frac{\text{M}\sin\theta}{\text{A}}\cdot(x^2 + y^2 - 2xy\cos\theta)\,dx\,dy,$$

$$\therefore \text{ mom.} = \frac{\text{M}.\sin\theta}{\text{A}}.\int_x\int_y (x^2 + y^2 - 2xy\cos\theta),$$

$$= \frac{\text{M}.\sin\theta}{\text{A}}.\int_x\left(x^2y + \frac{1}{3}\cdot y^3 - xy^2\cos\theta,\right)\text{from } y = 0, \text{ to } y = \frac{ax}{b},$$

$$= \frac{\text{M}.\sin\theta}{\text{A}}\cdot\left(\frac{a}{b} + \frac{1}{3}\cdot\frac{a^3}{b^3} - \frac{a^2}{b^2}\cdot\cos\theta\right).\int_x x^3, \text{from } x = 0, \text{ to } x = b$$

$$= \frac{\text{M}.\sin\theta}{4\text{A}}\cdot\left(\frac{a}{b} + \frac{1}{3}\cdot\frac{a^3}{b^3} - \frac{a^2}{b^2}\cdot\cos\theta\right).\,b^4,$$

$$= \text{M}.\frac{1}{6}\,(3b^2 + a^2 - 3ab\cos\theta),\ \because\ 2\text{A} = ab\sin\theta,$$

$$= \text{M}.\frac{1}{12}\,(6b^2 + 2a^2 - 3a^2 - 3b^2 + 3c^2),$$

$$= \text{M}.\frac{1}{12}\,(3.\overline{b^2 + c^2} - a^2).$$

Ex. 5. To find the moment of inertia of a triangle about its centre of gravity.

Let M be its mass; G its centre of gravity (Fig. 44); then, by the last example,

$$\text{mom. of } \triangle \text{ AGB about G} = \frac{M}{36} \cdot (3.\,\overline{BG^2 + AG^2} - c^2),$$

$$\ldots\ldots \text{AGC} \ldots\ldots = \frac{M}{36} \cdot (3.\,\overline{AG^2 + CG^2} - b^2),$$

$$\ldots\ldots \text{BGC} \ldots\ldots = \frac{M}{36} \cdot (3.\,\overline{BG^2 + CG^2} - a^2);$$

∴ by adding, we have mom. of the whole triangle

$$= \frac{M}{36} \cdot (6.\,\overline{AG^2 + BG^2 + CG^2} - a^2 - b^2 - c^2),$$

$$= \frac{M}{36} \cdot (2.\,\overline{a^2 + b^2 + c^2} - a^2 - b^2 - c^2),$$

$$= M . \frac{a^2 + b^2 + c^2}{36}.$$

Ex. 6. To find the moment of inertia of a circle about an axis, perpendicular to its plane, through its centre.

a, M, the radius and mass of the circle.

Then the mass of a circular annulus whose radii are r, and $r + \delta r = M . \dfrac{2r\delta r}{a^2}$, ultimately.

$$\therefore \text{mom. of inert. of this element} = M . \frac{2r^3 \delta r}{a^2},$$

$$\therefore \text{mom. of the circle} = M . \int_{r=a} \frac{2r^3}{a^2} = M \cdot \frac{a^2}{2}.$$

157. The moment of inertia of any plane figure round an axis, perpendicular to its plane, through the origin of co-ordinates = mom. round axis of x + mom. round axis of y.

For let μ = mass of a unit of the figure,

$$\therefore \text{mass of an element} = \mu \delta x\, \delta y,$$

$$\text{and its mom. round axis of } z = \mu \delta x\, \delta y\, (x^2 + y^2),$$

$$\therefore \text{mom. of the fig. round axis of } z = \mu \int_x \int_y y^2 + \mu \int_x \int_y x^2,$$

$$= \text{mom. round axis of } x + \text{that round axis of } y.$$

Ex. 7. To find the moment of inertia of a circle about its diameter.

$$\begin{aligned} \mathrm{M}\,.\,\frac{a^2}{2} &= \text{mom. round axis of } z, \\ &= \text{mom. about } x + \text{that about } y, \\ &= 2\,.\,(\text{mom. about } x)\,; \end{aligned}$$

$$\therefore \text{ mom. about } x = \mathrm{M}\,.\,\frac{a^2}{4}\,.$$

Ex. 8. To find the moment of inertia of an ellipse about its major axis.

Let a, b, be the semi-axes, xy any point within the ellipse; M = its mass, and μ = the mass of a unit; then, if $y' = \frac{a}{b}\cdot y$, xy' will be the corresponding point in the circumscribing circle, and the mom. of the ellipse

$$\begin{aligned} &= \mu \int_x \int_y y^2 = \mu\cdot\frac{b^3}{a^3}\cdot\int_x\int_{y'} y'^2, \\ &= \frac{b^3}{a^3}\cdot \text{ mom. of the circumscribing circle,} \\ &= \frac{b^3}{a^3}\cdot\left(\frac{a}{b}\cdot\mathrm{M}\right)\cdot\frac{a^2}{4} = \mathrm{M}\,.\,\frac{b^2}{4}\,. \end{aligned}$$

The mom. about axis of z = $\mathrm{M}.\ \dfrac{a^2+b^2}{4}$.

Ex. 9. To find the moment of inertia of a cone about its axis.

a = its altitude, b = radius of its base. Then the radius of a section ‖ to the base at distance x from the vertex $= \dfrac{bx}{a}$. Let M = mass of the cone. Then the mass of an elementary slice of the cone, whose thickness is δx, is ultimately

$$= 3\mathrm{M}\,.\,\frac{x^2\,\delta x}{a^3},$$

$$\therefore \text{ the mom. of this element} = 3\mathrm{M}\,.\,\frac{x^2}{a^2}\cdot\frac{b^2x^2\,\delta x}{2a^3} \text{ (Ex. 6).}$$

$$\therefore \text{ mom. of the cone} = \frac{3}{2}\cdot\mathrm{M}.\int_{x=a}\frac{b^2x^4}{a^5} = \mathrm{M}\,.\,\frac{3}{10}\,b^2.$$

Ex. 10. To find the moment of inertia of a sphere about a diameter.

Let this diameter be the axis of x.

An elementary slice of the sphere, whose thickness is δx, $= \mu\pi y^2 \delta x$, ultimately;

and the mom. of this element $= \mu\pi \, . \, y^2 \delta x \, . \, \frac{y^2}{2}$ (Ex. 6),

$$\therefore \text{ mom. of sphere } = \tfrac{1}{2}\mu\pi \, . \int_{x=\pm a} y^4,$$

$$= \tfrac{1}{2}\mu\pi \, . \int_{x=\pm a} (a^2-x^2)^2 = \text{M} \, . \, \frac{2}{5} a^2.$$

Ex. 11. To find the moment of inertia of a spherical shell, whose external and internal radii are a, b, about a diameter.

$$\text{Mom. of sphere radius } a = \mu \, . \, \frac{4}{3} \pi a^3 . \, \frac{2}{5} a^2 = \frac{8}{15} \mu\pi a^5,$$

$$. \; . \; . \; . \; . \quad \text{radius } b = \frac{8}{15} \mu\pi \, b^5,$$

$$\therefore \text{ mom. of the shell } = \frac{8}{15} \mu\pi \, . \, (a^5-b^5),$$

$$= \frac{2}{5} \text{ M} \, . \, \frac{a^5-b^5}{a^3-b^3}.$$

Ex. 12. To find the moment of inertia of an anchor ring, about an axis through its centre perpendicular to its plane.

Let D be the centre of the circle AB (Fig. 45), by the revolution of which, about the axis C, the anchor ring is formed.

$a =$ CD, $b =$ AD, $x \, y$ co-ordinates of a point in the circle, considering D as the origin, and DC as axis of x. Then, an element of the generating circle $= \mu \, y \, \delta x$,

and of the ring generated by it $= 2\pi\mu y \, (a-x) \, \delta x$,

$\therefore$ mom. of this element about C $= 2\pi\mu y \, (a-x)^3 \delta x$,

$\therefore$ mom. of the ring $= 2\pi\mu \int_{x=\pm b} y \, (a^3-3a^2 x + 3ax^2-x^3).$

Now $\mu\int_{x=\pm b} ya^3 = \mu a^3.$(area of the circle) $= \mu a^3 b^2\pi$,

$\mu\int_{x=\pm b} 3a^2yx = \mu.3a^2.b^2\pi.$(distance of the centre of gravity of the circle from D) $= 0$,

$\mu\int_{x=\pm b} 3ayx^2 = 3a.$(moment of inertia of the circle about its diameter) $= 3a.\mu b^2\pi.\frac{b^2}{4}$,

$$= \frac{3}{4}.\mu\pi.ab^4,$$

$2\pi\mu\int_{x=\pm b} yx^3 =$ (mom. of inert. of a sphere generated by the circle) — (mom. of inert. of an equal sphere). $= 0$;

∴ mom. of inert. of the ring

$$= 2\pi^2\mu ab^2\left(a^2 + \frac{3}{4}b^2\right),$$

$$= \text{M}\left(a^2 + \frac{3}{4}b^2\right).$$

Ex. 13. To find the moment of inertia of an ellipsoid about one of its principal axes.

The equation of the surface is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and an element of the mass $= \mu\delta x\,\delta y\,\delta z$; and the moment of this element about $a = \mu\delta x\,\delta y\,\delta z\,(y^2 + z^2)$; ∴ moment of the ellipsoid $= \mu\int_x\int_y\int_z(y^2 + z^2)$.

Now $\int_x\int_y\int_z y^2 = \int_x\int_y y^2z + \text{C} = 2c\int_x\int_y y^2\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$,

between $y = \mp\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$;

$= 2ac\int_y y^2.\left(\text{circular area, abscissa}\frac{x}{a}, \text{ and rad.}\sqrt{1 - \frac{y^2}{b^2}}\right) + \text{C}$

$= ac\pi\int_y y^2.\left(1 - \frac{y^2}{b^2}\right)$, between $\frac{x}{a} = \mp\sqrt{1 - \frac{y^2}{b^2}}$,

$= abc\pi\left(\frac{y^3}{3b} - \frac{y^5}{5b^3}\right) + \text{C} = \frac{4}{15}abc\pi.b^2$, between $y = \mp b$.

∴ mom. of inert. $= \mu.\frac{4}{15}abc\pi.(b^2 + c^2) = \text{M}.\frac{b^2 + c^2}{5}$. See Airy's Tracts, *Precession and Nutation.*

PROPERTIES OF THE PRINCIPAL AXES.

158. Given the moments of inertia of a body about three rectangular axes; to find its moment of inertia round any other axis passing through the origin.

Let ox', oy', oz' (Fig. 46) be the three rectangular axes, oI the new axis of rotation; $m =$ the mass of an element, $m\mathrm{I} \perp o\mathrm{I}$; $x'y'z'$ co-ordinates of m, $\alpha\,\beta\,\gamma$ the $\angle^s$ which oI makes with the co-ordinate axes. Q = the moment of inertia round oI.

$$\therefore\ o\mathrm{I} = x' \cos\alpha + y' \cos\beta + z' \cos\gamma,$$

$$\therefore\ (m\mathrm{I})^2 = x'^2 + y'^2 + z'^2 - (x' \cos\alpha + y' \cos\beta + z' \cos\gamma)^2.$$

$$\therefore\ \mathrm{Q} = \Sigma\, m\, (m\mathrm{I})^2,$$

$$= \Sigma\,(x'^2 m)\,.\,\sin^2\alpha + \Sigma(y'^2 m)\,.\,\sin^2\beta + \Sigma(z'^2 m)\,.\,\sin^2\gamma$$
$$-2\Sigma(y'z'm)\cos\beta\cos\gamma - 2\Sigma(x'z'm)\cos\alpha\cos\gamma - 2\Sigma(x'y'm)\cos\alpha\cos\beta.$$

Let $f = \Sigma\,(x'^2 m)$, $g = \Sigma\,(y'^2 m)$, $h = \Sigma\,(z'^2 m)$,

$\mathrm{F} = \Sigma\,(y'z'm)$, $\mathrm{G} = \Sigma\,(x'z'm)$ and $\mathrm{H} = \Sigma\,(x'y'm)$,

$$\therefore\ \mathrm{Q} = f\sin^2\alpha + g\sin^2\beta + h\sin^2\gamma - 2\mathrm{F}\cos\beta\cos\gamma - 2\mathrm{G}\cos\alpha\cos\gamma - 2\mathrm{H}\cos\alpha\cos\beta.$$

159. Def. Three rectangular axes, passing through the centre of gravity, such that $\Sigma\,(yzm) = 0$, $\Sigma\,(xzm) = 0$, and $\Sigma\,(xym) = 0$, are called the *principal axes.*

160. To shew that every body has three principal axes.

Suppose ox, oy, oz the three principal axes, O being the centre of gravity. Let $a\,a'\,a''$, $b\,b'\,b''$, $c\,c'\,c''$ be the cosines of the $\angle^s$ which ox, oy, oz respectively make with the axes of $x'\,y'\,z'$; $\alpha'\,\beta'\,\gamma'$ the $\angle^s$ which OI makes with $x\,y\,z$; $\mathrm{X} = \Sigma\,(x^2 m)$, $\mathrm{Y} = \Sigma\,(y^2 m)$ and $\mathrm{Z} = \Sigma\,(z^2 m)$.

$$\therefore\ Q = X \sin^2 \alpha' + Y \sin^2 \beta' + Z \sin^2 \gamma' \ldots \text{(Art. 158)},$$

and

$$X + Y + Z = \Sigma m(x^2 + y^2 + z^2) = \Sigma m(x'^2 + y'^2 + z'^2)$$
$$= f + g + h;$$
$$\therefore f + g + h - Q = X \cos^2 \alpha' + Y \cos^2 \beta' + Z \cos^2 \gamma',$$
$$= X(a \cos\alpha + a' \cos\beta + a'' \cos\gamma)^2 + Y(b \cos\alpha + b' \cos\beta + b'' \cos\gamma)^2$$
$$+ Z(c \cos\alpha + c' \cos\beta + c'' \cos\gamma)^2.$$

But

$$f + g + h - Q = f \cos^2\alpha + g \cos^2\beta + h \cos^2\gamma + 2F \cos\beta \cos\gamma$$
$$+ 2G \cos\alpha \cos\gamma + 2H \cos\alpha \cos\beta \ldots \text{(Art. 158)}.$$

Hence, by comparing the coefficients of like cosines in these two expressions, we have

$$\left.\begin{aligned} f &= a^2 X + b^2 Y + c^2 Z \\ g &= a'^2 X + b'^2 Y + c'^2 Z \\ h &= a''^2 X + b''^2 Y + c''^2 Z \end{aligned}\right\} \text{ and } \left\{\begin{aligned} F &= a'a''X + b'b''Y + c'c''Z \\ G &= aa''X + bb''Y + cc''Z \\ H &= aa'X + bb'Y + cc'Z \end{aligned}\right.$$

$$\therefore \left.\begin{aligned} af + a''G + a'H &= aX \\ a'g + aH + a''F &= a'X \\ a''h + a'F + aG &= a''X \end{aligned}\right\} \ldots (\alpha),$$

determining a', a'' from the first and second of these equations, and substituting in the third, we obtain

$$(X-f)(X-g)(X-h) - \{F^2(X-f) + G^2(X-g) + H^2(X-h)\}$$
$$= 2FGH \ldots. (1),$$

which being a cubic equation, has necessarily one possible value of X; substituting this value in (α), and observing that $a^2 + a'^2 + a''^2 = 1$, we have the values of $a\ a'\ a''$, which are all possible; and hence the position of the axis of x is determined. But we obtain the same final equation as (1), with Y or Z instead of X, when we seek to determine Y or Z; and ∴ X Y Z are the three roots of the cubic equation (1). If ∴ Y is not possible, let it $= R + S\sqrt{-1}$, and ∴ $Z = R - S\sqrt{-1}$; similar equations to (α) joined with $b^2 + b'^2 + b''^2 = 1$, and $c^2 + c'^2 + c''^2 = 1$, will enable us to determine $b = r + s\sqrt{-1}$, $b' = r' + s'\sqrt{-1}$, $b'' = r'' + s''\sqrt{-1}$; $c = r - s\sqrt{-1}$, $c' = r' - s'\sqrt{-1}$, $c'' = r'' - s''\sqrt{-1}$.

$$\therefore 0 = bc + b'c' + b''c'' = r^2 + s^2 + r'^2 + s'^2 + r''^2 + s''^2,$$

which is impossible, $\because$ $r^2, s^2, \ldots$ are essentially positive, and $\therefore$ Y Z are possible, and $\therefore$ the body has three principal axes.

161. Given one principal axis to find the other two.

Let the given axis be that of z'; then c, c', a'', b'' are each $= 0$, and $c'' = 1$; and if θ be the $\angle$ which the axis of x makes with that of x', then $a = \cos\theta$, $a' = \sin\theta$, and from equations (α),

$$f\cos\theta - \mathrm{H}\sin\theta = \mathrm{X}\cos\theta,$$
$$g\sin\theta - \mathrm{H}\cos\theta = \mathrm{X}\sin\theta,$$

and $\therefore$ by eliminating X,

$$\frac{2\mathrm{H}}{g-f} = \tan 2\theta = \tan(180^\circ + 2\theta),$$

from which the positions of the axes of x and y are known.

162. Given A, B, C, the moments about the principal axes, to find the moment about an axis, through the centre of gravity, inclined at given $\angle^{s}$ α, β, γ, to them.

If Q be the required moment of inertia, we have Q

$$= \sin^2\alpha \,.\, \Sigma(x^2m) + \sin^2\beta \,.\, \Sigma(y^2m) + \sin^2\gamma \,.\, \Sigma(z^2m),$$
$$= (\cos^2\beta + \cos^2\gamma) \,.\, \Sigma(x^2m) + (\cos^2\alpha + \cos^2\gamma)\,\Sigma(y^2m) + (\cos^2\alpha + \cos^2\beta)\,\Sigma(z^2m),$$
$$= \cos^2\alpha \,.\, \Sigma m(y^2+z^2) + \cos^2\beta \,.\, \Sigma m(x^2+z^2) + \cos^2\gamma \,.\, \Sigma m(x^2+y^2),$$
$$= \mathrm{A}\cos^2\alpha + \mathrm{B}\cos^2\beta + \mathrm{C}\cos^2\gamma.$$

163. The sum of the moments of a body about any three rectangular axes, through the centre of gravity, is a constant quantity ($=$ A + B + C).

Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, be the $\angle^{s}$ which the three axes respectively make with the principal axes; then the sum of the moments, by last Article,

$$= \mathrm{A}\cos^2\alpha + \mathrm{B}\cos^2\beta + \mathrm{C}\cos^2\gamma,$$
$$+ \mathrm{A}\cos^2\alpha' + \mathrm{B}\cos^2\beta' + \mathrm{C}\cos^2\gamma',$$
$$+ \mathrm{A}\cos^2\alpha'' + \mathrm{B}\cos^2\beta'' + \mathrm{C}\cos^2\gamma'',$$
$$= \mathrm{A} + \mathrm{B} + \mathrm{C}.$$

164. The moment about one of the principal axes is a maximum, and about another a minimum.

For $Q = A \cos^2\alpha + B \cos^2\beta + C \cos^2\gamma$,

but $Q = Q \cos^2\alpha + Q \cos^2\beta + Q \cos^2\gamma$,

$\therefore 0 = (A - Q)\cos^2\alpha + (B - Q)\cos^2\beta + (C - Q)\cos^2\gamma$;

and, consequently, Q cannot be either greater or less than each of the three quantities A, B, C; and ∴ it always lies between the greatest and least of those quantities, and ∴ one of them is a maximum and one a minimum.

Cor. The moment of inertia about one of the principal axes is less than about any other axis whatever, either within or without the body (Art. 155).

165. In Art. 157 it was shewn, that if the figure be a physical plane, $C = A + B$, and hence the moment of inertia of any plane figure about an axis through its centre of gravity, the projection of which on its plane makes $\angle^s\ \alpha, \beta$, with the axes of x, y,

$$= A \cos^2\alpha + B \cos^2\beta + (A + B)\cos^2\gamma,$$
$$= A \sin^2\beta + B \sin^2\alpha.$$

Cor. 1. If the moments (A, B) about two of the principal axes are equal, the moment about any other axis

$$= A \cos^2\alpha + B \cos^2\beta + C \cos^2\gamma,$$
$$= A(\cos^2\alpha + \cos^2\beta) + C \cos^2\gamma,$$
$$= A \sin^2\gamma + C \cos^2\gamma,$$

which is therefore dependent only on its inclination to the axis of z.

Cor. 2. The moment of inertia of any plane figure round an axis in its own plane

$$= A \cos^2\alpha + B \cos^2\beta,$$
$$= A \cos^2\alpha + B \sin^2\alpha,$$

and hence, if the moments about the two principal axes be equal, the moments about all axes in that plane will be the same.

Cor. 3. If the moments about the three principal axes are equal, the moment about every axis through the centre of gravity will be the same.

$$\begin{aligned} \text{For } Q &= A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma, \\ &= A (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma), \\ &= A. \end{aligned}$$

166. To find the axes of rotation, passing through the centre of gravity, for which the moments of inertia are the same.

Here Q is constant, and from Art. 162, if x, y, z be the coordinates of a point in one of the axes of rotation at the distance a from the origin, and α, β, γ be the same as before,

$$\begin{aligned} a^2 Q &= A \,.\, a^2 \cos^2 \alpha + B \,.\, a^2 \cos^2 \beta + C \,.\, a^2 \cos^2 \gamma, \\ &= Ax^2 + By^2 + Cz^2 \;\ldots\ldots\; (1), \\ \text{but } a^2 &= x^2 + y^2 + z^2 \;\ldots\ldots\ldots\; (2). \end{aligned}$$

Hence the required axes pass through points which are the locus of the intersection of a sphere, and an ellipsoid, whose equations are (2) (1).

The following example is added to illustrate the general formula of Art. 162.

Ex. To find the moment of inertia of an ellipsoid about any axis passing through its centre of gravity.

Let a, b, c be the semi-axes; A, B, C the moments about them; α, β, γ the angles at which the axis of rotation is inclined to them; M = the mass of the ellipsoid.

$\therefore A = \frac{1}{5} M \,.\, (b^2 + c^2), B = \frac{1}{5} M (a^2 + c^2), C = \frac{1}{5} M (a^2 + b^2) \,.\,.$ (Ex. 13, Art. 157).

$$\therefore \; Q = \tfrac{1}{5} \,.\, M (a^2 \sin^2 \alpha + b^2 \sin^2 \beta + c^2 \sin^2 \gamma).$$

CHAPTER IX.

ON THE APPLICATION OF THE GENERAL PRINCIPLES OF CHAPTER VII.

Ex. 1. Two weights P, Q, connected by a flexible string without weight, hang over a fixed pulley, and P begins to descend; to determine the motion.

Let P, Q denote the masses of the weights; a the radius, m the mass, and mk^2 the moment of inertia of the pulley.

Then the pulley resists the motion as much as $m\,.\,\frac{k^2}{a^2}$ (Art. 146) placed at its circumference, $\therefore$ the mass resisting motion $= \mathrm{P} + \mathrm{Q} + m\,.\,\frac{k^2}{a^2}$, and the moving force of $\mathrm{P} = \mathrm{P}g$, of $\mathrm{Q} = -\,\mathrm{Q}g$,

$\therefore$ accelerating force on $\mathrm{P} = \dfrac{(\mathrm{P}-\mathrm{Q})\,g}{\mathrm{P}+\mathrm{Q}+m\,.\,\frac{k^2}{a^2}}$ . . . (Art. 16),

which being constant, the time and space may be determined by the formulæ of Art. 53.

Ex. 2. The weight P hangs over a wheel, and Q over the axle; to determine the motion, P descending.

a, b, are the radii of the wheel and axle, m their mass, and mk^2 their moment of inertia. Then the weight Q, and the wheel and axle resist the motion as much as $\mathrm{Q}\,\frac{b^2}{a^2}$, $m\,\frac{k^2}{a^2}$, placed at the circumference of the wheel; and the moving force acting at that circumference $= \left(\mathrm{P} - \mathrm{Q}\,.\,\frac{b}{a}\right).g$;

$$\therefore \text{ the accelerating force on P} = \frac{\left(\text{P} - \text{Q}\cdot\frac{b}{a}\right)g}{\text{P} + \text{Q}\cdot\frac{b^2}{a^2} + m\frac{k^2}{a^2}},$$

$$= \frac{a\,(\text{P}a - \text{Q}b)\cdot g}{\text{P}a^2 + \text{Q}b^2 + mk^2} \dots (1),$$

which is constant.

In this example, having given the radius of the axle, to find that of the wheel, that P may draw Q through a given space in the least time possible.

$(\text{time})^2 \propto \frac{\text{space}}{\text{force}}$ (Art. 53) $\propto \frac{1}{\text{force}}$, $\therefore$ the force must be a maximum from the variation of a; hence, by putting the differential coefficient of (1) $= 0$, we have

$$\left(\frac{a}{b}\right)^2 + 2\left(1 + \frac{mk^2}{\text{Q}b^2}\right)\cdot\left(\frac{a}{b}\right) = \frac{\text{Q}}{\text{P}} + \frac{mk^2}{\text{P}b^2},$$

from which equation a is to be determined.

Ex. 3. The axis of a horizontal cylinder passes through the centre of gravity of a given mass, and a weight hanging from a string which is wrapped round the cylinder begins to descend, communicating an angular motion to the mass; to determine the motion of the weight.

Let m = the given mass, mk^2 its moment of inertia; P = the mass of the weight, a = the radius of the cylinder. Then, as before, the moving force = Pg, and the mass resisting motion is equivalent to $\text{P}+m\cdot\frac{k^2}{a^2}$ at the circumference of the cylinder;

$$\therefore \text{ the accelerating force on P} = \frac{\text{P}g}{\text{P} + m\cdot\frac{k^2}{a^2}} = \frac{\text{P}a^2g}{\text{P}a^2 + mk^2},$$

which is constant.

In this example, to determine the radius of the cylinder, that the angular velocity generated in a given time may be a maximum.

The velocity of P = (force). (time) by Art. 50 $\propto$ force,

$\therefore$ the $\angle^r$ velocity $\propto \frac{\text{force}}{a} \propto \frac{\mathrm{P}a}{\mathrm{P}a^2 + mk^2}$, a maximum when $\mathrm{P}a + \frac{mk^2}{a}$ is a minimum, $\therefore a = k\sqrt{\frac{m}{\mathrm{P}}}$, *by Diff. Calc.*

Ex. 4. A string is wrapped round a cylinder, and one end of it fastened at the top of an inclined plane, the cylinder then begins to descend, unwinding the string; to determine the motion.

α = the inclination of the plane to the horizon, a = the radius of the cylinder, m = its mass, mk^2 = its moment of inertia about its axis, T = the tension of the string.

$\therefore$ accelerating force on the centre of gravity, down the plane

$$= g \sin \alpha - \frac{\mathrm{T}}{m} \ldots . \text{ (Art. 137)},$$

and accelerating force on the circumference,

$$= \frac{\mathrm{T}}{m} \cdot \frac{a^2}{k^2} \ldots\ldots \text{ (Arts. 146, 137, 16)};$$

and by the nature of the machine, the centre of gravity and the circumference move with the same velocities, and $\therefore$ they are acted on by the same accelerating forces,

$$\therefore \frac{\mathrm{T}}{m} \cdot \frac{a^2}{k^2} = g \sin \alpha - \frac{\mathrm{T}}{m}.$$

$$\therefore \frac{\mathrm{T}}{m} = \frac{k^2 g \sin \alpha}{a^2 + k^2}.$$

$\therefore$ the accel. force down the plane $= \frac{a^2 g \sin \alpha}{a^2 + k^2}$, which is constant. This is the force which accelerates the motion of a cylinder *rolling* down a plane, for the friction supplies the place of the string.

Ex. 5. A body of any form, suspended in a horizontal position from a fixed axis, is acted on by gravity, and begins to oscillate in a vertical plane; to determine the motion.

Let S be the centre of suspension, G the centre of gravity, θ the $\angle$ which SG makes with the horizon; $a =$ SG, $m =$ the mass, mk^2 its momentum of inertia about G.

Then the $\angle^r$ velocity $= d_t\theta$,

$$\therefore \text{ the } vis\ viva = mk^2 . d_t\theta^2 + ma^2 d_t\theta^2 \ . \ . \ . \text{ (Art. 140).}$$

But if G descend vertically through the same space ($a \sin \theta$), its *vis viva* would $= m . 2ag \sin \theta$, (Art. 53), and $\therefore$ by the principle of the conservation of *vis viva* (Art. 138),

$$m(k^2 + a^2) . d_t\theta^2 = m . 2ag \sin \theta,$$

$$\therefore d_t\theta^2 = \frac{2ag \sin \theta}{k^2 + a^2}.$$

Problems respecting the motion of any combination of machines, may also very readily be solved by Art. 144.

Ex. 6. A heavy chain is wrapped round a horizontal cylinder, and one end hanging down begins to descend, communicating a rotatory motion to the cylinder; to determine the motion.

$a =$ the length of the chain, m its mass, c the part hanging down at first, x the part at the time t; b the radius of the cylinder, M its mass, and $\mathrm{M}k^2$ its moment of inertia.

Then the mass of the part hanging down $= m . \frac{x}{a}$,

$$\text{and } \therefore \text{ the moving force} = m . \frac{x}{a} . g.$$

The cylinder resists the motion as much as $\mathrm{M} \cdot \frac{k^2}{b^2}$ placed at its circumference;

$$\therefore \text{ the mass moved} = m + \mathrm{M} \cdot \frac{k^2}{b^2},$$

$$\therefore \text{ the accelerating force on the end of the chain} = \frac{\frac{mx}{a} \cdot g}{m + \mathrm{M} \cdot \frac{k^2}{b^2}}.$$

Hence, if v be the velocity of the extremity of the chain,

$$v d_x v = \frac{b^2}{a} \cdot \frac{mxg}{mb^2 + Mk^2},$$

$$\therefore v^2 = \frac{b^2}{a} \cdot \frac{mg(x^2 - c^2)}{mb^2 + Mk^2},$$

$$\therefore t = \sqrt{\frac{a}{g} \cdot \left(1 + \frac{Mk^2}{mb^2}\right)} \cdot \log_e\left(\frac{x}{c} + \sqrt{\frac{x^2}{c^2} - 1}\right) \ldots (\text{Art. } 77).$$

Ex. 7. A uniform chain is laid upon a curve, situated in a vertical plane, and begins to slide; to determine the motion.

Let ACB (Fig. 47) be the chain, C the highest point, Cab vertical, Aa, Bb horizontal. Cab the axis of x, C being the origin of co-ordinates; s = the length of any portion CP of the chain, xy the co-ordinates of P; a = the length of the whole chain, m = its mass.

$\therefore$ the mass of an element at P $= \dfrac{m\delta s}{a} = \dfrac{m}{a} . d_x s \, \delta x$ ultimately,

$$\text{and } \therefore \text{ its moving force} = \frac{mg}{a} . \delta x,$$

$$\text{and } \therefore \text{ the moving force of the chain} = \frac{mg}{a} . x + C,$$

$$= \frac{mg}{a} . ab,$$

= the weight of a part of the chain whose length is equal to ab.

$$\therefore \text{ the accelerating force on the chain} = \frac{ab}{ACB} \cdot g.$$

167. Ex. 8. A free body at rest, receives a sudden impulse at a given point; to determinate the *axis of spontaneous rotation*, or that axis about which it will begin to revolve.

Let the plane of the paper pass through G (Fig. 48), the centre of gravity of the body and the point A at which the impulse F is communicated in the direction Aa $\perp$ AG; m = the mass of the body, mk^2 = its moment of inertia about G. Then supposing F applied at G (Art. 137), it will, in a certain time, cause the body to move in such a position, that G, A, shall be at g, a; and, then, supposing the centre of gravity fixed at g, and F applied at a (Art. 137), the body will, in the same time as before, describe the $\angle\,Bgb$, and $\therefore$ Bg will be the true position of the body at the end of the given time, and $\therefore$ B ultimately remains at rest, as also every point in a line through B $\perp$ to the plane of the paper, which line is $\therefore$ the axis required. Now $Bb = Gg$, that is, the velocity of B from rotation = that from translation, and $\therefore$ the forces which produce them are equal. $\frac{F}{m}$ is the force producing translation (Art. 16), and $\frac{F\,.\,AG}{mk^2}\cdot GB$ is the force producing the rotation of B (Art. 144).

$$\therefore\ \frac{F}{m}\cdot\frac{AG\,.\,GB}{k^2} = \frac{F}{m},\ \text{and}\ \therefore\ AG\,.\,GB = k^2.$$

Hence (Art. 149) if A be considered the centre of oscillation, B will be the centre of suspension.

Ex. 9. A uniform heavy rod slides between a vertical and horizontal line; to determine the motion.

Let G (Fig. 49) be the centre of gravity of the rod, sliding between the vertical and horizontal lines CA, CB; $a = CG = AG = BG$, $\theta = ACG$. Then the velocity of G $= a d_t\theta$; and if m be the mass, $\frac{1}{3}ma^2$ = the moment of inertia of the rod,

$$\text{and}\ \therefore\ \text{the } vis\ viva = ma^2 d_t\theta^2 + \frac{1}{3}ma^2 d_t\theta^2 \ldots \text{(Art. 140)}$$

$$= \frac{4}{3}\cdot ma^2 d_t\theta^2.$$

But if the particles of the rod were all to fall freely through the same vertical spaces as before, the *vis viva* acquired would be $m\,.\,2ag\,(\cos\alpha - \cos\theta)$, α being the value of θ when the motion begins,

$$\therefore \frac{4}{3} m a^2 d_t \theta^2 = m \,.\, 2ag \,(\cos \alpha - \cos \theta) \ldots \text{(Art. 138)},$$

$$\therefore t = \sqrt{\frac{2a}{3g}} \,.\, \int_\theta^{\cdot} \frac{1}{\sqrt{\cos \alpha - \cos \theta}} \,.$$

Ex. 10. A heavy rod, with one end resting upon a smooth horizontal plane, is suffered to fall in a vertical plane; to determine the motion.

Let AB (Fig. 50) be the rod, G its centre of gravity, m its mass, mk^2 its moment of inertia, BO the horizontal plane, GO vertical, $b =$ BG, $\theta =$ OGB. Then ∵ there is no horizontal force, G descends in the vertical line GO, and its velocity $= d_t \,(b \cos \theta) = - b \sin \theta \,.\, d_t \theta$,

$$\therefore \text{the } \textit{vis viva} = m b^2 \sin^2 \theta \,.\, d_t \theta^2 + m k^2 .\, d_t \theta^2.$$

But if its particles had fallen freely through the same vertical spaces, it would have acquired a *vis viva* $= m \,.\, 2bg$ $(\cos \alpha - \cos \theta)$.

$\therefore m(k^2 + b^2 \sin^2 \theta) \,.\, d_t \theta^2 = m \,.\, 2bg \,(\cos \alpha + \cos \theta) \ldots .$ (Art. 138),

$$\therefore t = \int_\theta^{\cdot} \cdot \left(\frac{k^2 + b^2 \sin^2 \theta}{2bg\,(\cos \alpha - \cos \theta)} \right)^{\frac{1}{2}}.$$

Cor. Since $\frac{\text{AM}^2}{\text{AB}^2} + \frac{\text{MO}^2}{\text{AG}^2} = \cos^2 \theta + \sin^2 \theta = 1$, ∴ the locus of A will be an ellipse, whose centre is O, and semi-axes are AB, AG.

Ex. 11. A rod slides with its ends resting on two given inclined planes; to determine the motion, which is supposed to take place in a vertical plane perpendicular to the intersection of the two inclined planes.

Let the plane of the paper be that in which the motion takes place; AO, BO, OI, (Fig. 51) be the inclined planes and the horizon; G the centre of gravity of the rod AB, m its mass, mk^2 its moment of inertia, AG $= a$, BG $= b$; α, β, θ, the $\angle$s which AO, BO, AB, make with the horizon; xy the co-ordinates of G; O being the origin of co-ordinates, OI the axis of x $\parallel$ to the horizon.

The *vis viva* $= m\,(d_t x^2 + d_t y^2) + mk^2 d_t \theta^2 \ldots$ (Art. 140).

But if the particles of the rod had fallen freely through the same vertical spaces as before, its *vis viva* would have been $m\,.\,2g\,(h - y)$, h being the value of y when the motion commences.

$$\therefore\ d_t x^2 + d_t y^2 + k^2\, d_t\theta^2 = 2g\,(h - y) \ldots . \text{ (Art. 138).}$$

But, by the nature of the figure,

$$x\,.\sin(\alpha + \beta) = b\cos\beta\,.\sin(\alpha - \theta) - a\cos\alpha\,.\sin(\beta + \theta),$$
$$y\,.\sin(\alpha + \beta) = b\sin\beta\,.\sin(\alpha - \theta) + a\sin\alpha\,.\sin(\beta + \theta).$$

By eliminating $d_t x$, $d_t y$ from the above equation, by means of these, we obtain $d_t\theta^2 =$

$$\frac{\{C - a\sin\alpha\,.\sin(\beta+\theta) - b\sin\beta\,.\sin(\alpha-\theta)\}\,.\,2g\sin(\alpha+\beta)}{a^2\cos^2(\beta+\theta) - ab\,\{2\cos(\alpha+\beta)\,.\cos(\alpha-\theta)\,.\cos(\beta+\theta) - 1\} + b^2\cos^2(\alpha-\theta)}.$$

Ex. 12. Two rough balls, of given elasticity, moving uniformly in straight lines, impinge on each other; to find their motions after impact.

Let C be the point and Cx (Fig. 52) the plane of impact of the balls, whose masses are A, B; aA, bB their paths before impact; Aa', Bb' afterwards; a, b their velocities before impact; α, β the $\angle$s of incidence, λ the elasticity, X the force exerted on each other during impact $\parallel$ Cx, r the radius of A.

Their relative velocity before impact $\perp$ Cx

$$= a\cos\alpha + b\cos\beta,$$

and $\therefore$ (by Art. 28) A's velocity after impact $\perp$ Cx

$$= \frac{(1 + \lambda)\,B}{A + B}\,(a\cos\alpha + b\cos\beta) - a\cos\alpha \ldots\ldots (1),$$

also A's velocity after impact $\parallel$ Cx

$$= a\sin\alpha - \frac{X}{A} \ldots\ldots \text{(Arts. 137, 15.)} \ldots (2);$$

and since $\frac{2}{5}Ar^2$ = the moment of inertia of A, ∴ A resists $\angle^r$ motion as much as $\frac{2}{5}$ A placed at its circumference.

∴ velocity of A's circumference $= \frac{X}{\frac{2}{5}A}$, and ∴ of B's $= \frac{X}{\frac{2}{5}B}$.

Now, by reason of their roughness, there will be no sliding between the balls, and ∴ the velocities of the parts in contact will be the same for both.

But velocity of C in A $= a\sin\alpha - \frac{X}{A} - \frac{5X}{2A} = a\sin\alpha - \frac{7X}{2A}$,

. . . . in B $= b\sin\beta + \frac{X}{B} + \frac{5X}{2B} = b\sin\beta + \frac{7X}{2B}$,

which being equal, we have

$$a\sin\alpha - b\sin\beta = \frac{7}{2}\left(\frac{X}{A} + \frac{X}{B}\right),$$

and, by means of this, eliminating X from equation (2), we find,

A's velocity after impart ‖ Cx

$$= a\sin\alpha - \frac{2}{7}\cdot\frac{B}{A+B}(a\sin\alpha - b\sin\beta),$$

from this, and equation (1), all the circumstances of A's motion are known.

Ex. 13. A body being terminated at its lower extremity in a curve surface, rests on another curve surface, in a position of stable equilibrium, and is slightly disturbed; to find the time of *titubation, i.e.* of making a small oscillation.

Let the plane of the paper passing through G the centre of gravity of the body be the plane of titubation; Pc, PC the sections of the surfaces made by it; B, A their centres of curvature; c, C points which are in contact when the body is in a position of equilibrium; $\theta = \angle$ which Bc makes with AC produced. m, mk^2 the mass and moment of inertia about G of the titubating body; $R = AC$, $r = Bc$.

$$\therefore \angle BAC = \frac{r\theta}{R+r},$$

$$\therefore \text{ the moment of the forces about P} = mg\,.\left(\frac{\mathrm{R}r\theta}{\mathrm{R}+r} - \mathrm{G}c\,.\,\theta\right),$$

$$\text{and the moment of inertia about P} = m\,(k^2 + \mathrm{PG}^2),$$
$$= m\,(k^2 + \mathrm{G}c^2)\text{ ultimately},$$

$$\therefore\ d_t^2\theta = \frac{\dfrac{\mathrm{R}r}{\mathrm{R}+r} - \mathrm{G}c}{k^2 + \mathrm{G}c^2}\cdot g\theta\ \ldots\ldots\ (\text{Art. } 144),$$

and ∴ the length of the simple pendulum

$$= \frac{k^2 + \mathrm{G}c^2}{\dfrac{\mathrm{R}r}{\mathrm{R}+r} - \mathrm{G}c};$$ by *Integ. Cal.* and Art. 113, Cor.

Ex. 14. A rod of given mass revolves in a horizontal plane about a given point, and upon it is placed a ring; to determine the motion.

Let m be the mass of the ring, M that of the rod, $\mathrm{M}k^2$ its moment of inertia about the given point. Take that point for the origin of co-ordinates, and the plane of revolution for that of xy; x, y, r the co-ordinates and distance of m from the origin, θ the ∠ which r makes with the axis of x, and P = the pressure of the rod against the ring.

Then for the motion of the rod $d_t^2\theta = -\dfrac{\mathrm{P}r}{\mathrm{M}k^2}$ (Art. 144),

and for the motion of the ring $d_t^2x = -\dfrac{\mathrm{P}}{m}\cdot\dfrac{y}{r},\ d_t^2y = \dfrac{\mathrm{P}}{m}\cdot\dfrac{x}{r}$,

$$\therefore\ y\,d_t^2x - x\,d_t^2y = -\frac{\mathrm{P}r}{m} = \frac{\mathrm{M}}{m}\,.\,k^2d_t^2\theta,$$

$$\therefore\ y\,d_tx - x\,d_ty = \frac{\mathrm{M}}{m}\cdot k^2d_t\theta + \text{constant},$$

$$\text{or, } -r^2d_t\theta = \frac{\mathrm{M}}{m}\cdot k^2d_t\theta + \text{constant},$$

$$\therefore\ d_t\theta = \frac{\text{constant}}{mr^2 + \mathrm{M}k^2}\ \ldots\ (1).$$

Again, from the original equations,

$$d_tx\,d_t^2x + d_ty\,d_t^2y = \frac{\mathrm{P}}{mr}\cdot(x\,d_ty - y\,d_tx) = \frac{\mathrm{P}r\,d_t\theta}{m} = -\frac{\mathrm{M}}{m}.\,k^2d_t\theta\,d_t^2\theta,$$

$$\therefore\ d_t x^2 + d_t y^2 = \text{constant} - \frac{\mathrm{M}}{m} \cdot k^2 d_t \theta^2,$$

$$\text{or, } r^2 d_t \theta^2 + d_t r^2 + \frac{\mathrm{M}}{m} . k^2 d_t \theta^2 = \text{constant},$$

and by eliminating $d_t \theta$ by means of (1), we obtain

$$d_\theta r^2 = \mathrm{C}\left(r^2 + \frac{\mathrm{M}}{m} . k^2\right)^2 - \left(r^2 + \frac{\mathrm{M}}{m} . k^2\right),$$

which is the differential equation of the path of the ring.

Ex. 15. A body A (Fig. 54) revolves uniformly in the circumference of a circle, drawing after it another body m by means of a string; to determine the motion of m.

Let B the centre of the circle be the origin of co-ordinates, $x = \mathrm{BM}$, $y = m\mathrm{M}$, $a = \mathrm{AB}$, $b = \mathrm{A}m$, $\omega = \angle^r$ velocity of AB, $\theta =$ the $\angle$ which Am makes with BM, $\mathrm{T} =$ the tension of the string Am.

$$\therefore\ d_t^2 x = -\frac{\mathrm{T}}{m} . \cos\theta,\ d_t^2 y = \frac{\mathrm{T}}{m} . \sin\theta,$$

$$\therefore\ \sin\theta . d_t^2 x + \cos\theta . d_t^2 y = 0, \text{ by eliminating T.}$$

$$\text{But } x = a\cos\omega t + b\cos\theta,\ y = a\sin\omega t - b\sin\theta,$$

$$\therefore\ a\omega^2 . \sin(\omega t + \theta) + b d_t^2 \theta = 0, \text{ by substitution and reduction;}$$

$$\therefore\ a\omega^2 \sin(\omega t + \theta)\, d_t(\omega t + \theta) + b d_t(\omega t + \theta)\, d_t^2(\omega t + \theta) = 0,$$

$$\therefore\ -2a\omega^2 \cos(\omega t + \theta) + b d_t(\omega t + \theta)^2 = \text{constant};$$

$$\therefore\ \omega + d_t\theta = \left\{\mathrm{C} + \frac{2a\omega^2}{b} \cdot \cos(\omega t + \theta)\right\}^{\frac{1}{2}}.$$

168. Ex. 16. To find the time of vibration of a musical string.

Let ABW (Fig. 55) be the string, w its weight, a its length, W the weight which stretches it; P, Q two points in it very near each other; A B horizontal take for axis of x; Pp, Qq vertical. A P B the position of the string at the time t reckoning from the time when the string coincided with the horizontal line AB; $x = \mathrm{A}p$, $x + \delta x = \mathrm{A}q$, $y = \mathrm{P}p$,

$y + \delta y = \mathrm{Q}q$; $s = \mathrm{AP}$, $\mathrm{PQ} = \delta s$, and suppose the vibrations so small that APB = AB very nearly; and the tension the same in all the positions of the string.

The moving force on PQ in direction of y

$$= -\mathrm{W}d_s y + \mathrm{W}(d_s y + d_s \delta y),$$

$$= \mathrm{W}d_s \delta y = \mathrm{W}d_x{}^2 y \,.\, \delta s, \;\because\; \delta x = \delta s \text{ nearly};$$

and the mass of $\mathrm{PQ} = \dfrac{w\delta s}{ag}$,

$\therefore\ d_t^2 y =$ the accelerating force (Art. 80) on PQ

$$= \frac{\mathrm{W}ag}{w} \cdot d_x{}^2 y, \text{ (Art. 16)},$$

$$\therefore\ y = \mathrm{F}(x + bt) + f(x - bt),$$

by writing $b^2 w$ for $\mathrm{W}ag$, and integrating.

Now when $t = 0$, whatever be the value of x, $y = 0 = \mathrm{F}x + fx$,

$$\therefore fx = -\mathrm{F}x, \text{ and } \therefore f(x - bt) = -\mathrm{F}(x - bt),$$

$$\therefore y = \mathrm{F}(x + bt) - \mathrm{F}(x - bt).$$

Again, whatever be the value of t, when $x = 0$, $y = 0 = \mathrm{F}(bt) - \mathrm{F}(-bt)$, and $\therefore$ u being any variable quantity whatever, $\mathrm{F}u = \mathrm{F}(-u)$, $\therefore\ \mathrm{F}(x - bt) = \mathrm{F}(bt - x)$.

And when $x = a$, $y = 0 = \mathrm{F}(bt + a) - \mathrm{F}(bt - a)$, $\therefore\ \mathrm{F}(u + a) = \mathrm{F}(u - a)$.

Hence $\mathrm{F}u = \mathrm{F}(u \pm 2a) = \mathrm{F}(u \pm 4a) = \dots\dots\dots = \mathrm{F}(u \pm 4na)$, n being any integer.

$$\begin{aligned}\therefore\ y &= \mathrm{F}(bt + x) - \mathrm{F}(bt - x),\\ &= \mathrm{F}(bt + 2a + x) - \mathrm{F}(bt + 2a - x),\\ &= \mathrm{F}(bt + 4a + x) - \mathrm{F}(bt + 4a - x),\\ &= \dots\dots\dots\dots\dots\\ &= \mathrm{F}\left\{b\,.\left(t + \frac{2na}{b}\right) + x\right\} - \mathrm{F}\left\{b\left(t + \frac{2na}{b}\right) - x\right\}.\end{aligned}$$

Hence the point P will be in the same position at the times $t,\ t + \dfrac{2a}{b},\ t + \dfrac{4a}{b}, \dots\dots\ t + \dfrac{2na}{b}$; and $\therefore$ the time of a vibration $= \dfrac{2a}{b} = 2\sqrt{\left(\dfrac{w}{\mathrm{W}} \cdot \dfrac{a}{g}\right)}$.

169. Cor. To determine the form of the function F, in order that the vibrations may be symmetrical; that is, that all parts of the string may come to the horizontal line AB at the same instant.

In this case, for all values of x,

$$y = \mathrm{F}(bt + x) - \mathrm{F}(bt - x),$$

$$= 2 . \left\{\mathrm{F}'(bt) . x - \mathrm{F}'''(bt) . \frac{x^3}{1 . 2 . 3} + \mathrm{F}^{v}(bt) . \frac{x^5}{1 . 2 \ldots 5} - \ldots\right\}$$

$\therefore$ $\mathrm{F}'(bt)$, $\mathrm{F}'''(bt)$, $\mathrm{F}^{v}(bt)$ .... are each $= 0$, when $t = 0$, $\frac{a}{b}$, $\frac{2a}{b}$, ... which conditions will be satisfied, if $\mathrm{F}'(bt) \propto \sin\left(\frac{n\pi}{a} . bt\right)$, n being any integer whatever, and $\therefore$ $\mathrm{F}(bt) = -\mathrm{C}\cos\left(\frac{n\pi}{a} . bt\right)$.

Hence $y = -\mathrm{C}\left\{\cos\frac{n\pi}{a}(bt + x) - \cos\frac{n\pi}{a}(bt - x)\right\}$,

$$= 2\mathrm{C}\sin\frac{n\pi bt}{a} . \sin\frac{n\pi x}{a},$$

from which the form of the curve at any instant is known.

Since $y = 0$, whatever be the value of t, when $x = \frac{ma}{n}$, (m being any integer) the form of the curve may be similar to any one of the curves in Fig. 56, and the time will be

$$= \frac{2a}{nb} = \frac{2}{n}\sqrt{\left(\frac{w}{\mathrm{W}} . \frac{a}{g}\right)}.$$

170. Ex. 17. To find the velocity of sound in elastic solid bodies.

All solid bodies, such as glass, iron, stone, ... are in a certain degree elastic, and accordingly they are conductors of sound.

Let AP (Fig. 57) be a cylinder of the body of uniform thickness in a state of pulsation; A = the area of a section of it $\perp$ to its axis, AP = y, PQ = δy, ρ' the density of the cylinder in its natural state, ρ = the density at P; x, δx, the natural lengths of AP, PQ; p = the pressure on a unit of the section at P.

Then the mass of AP $= \rho' A x$, and of PQ $= \rho' A \delta x = \rho A \delta y$ ultimately.

$$\therefore\ d_x y = \text{limit of } \frac{\delta y}{\delta x} = \frac{\rho'}{\rho}.$$

But by the nature of elastic bodies $\rho\left(1 - \frac{p}{E}\right) = \rho'$, E being a constant, depending on the elasticity of the body, to be determined by experiment.

$$\therefore\ d_x y = 1 - \frac{p}{E}, \text{ and } \therefore\ d_x p = -\,E d^2_x y.$$

Now the pressure on the section through A $= Ap$, and on that through Q $= A\,(p + \delta p)$,

$\therefore$ the moving force on PQ $= Ap - A\,(p + \delta p) = -\,A\,.\,\delta p$,

and the mass of PQ $= A\rho'\,.\,\delta x$,

$\therefore$ the accelerating force on PQ

$$= \text{lim. } \frac{-A\,.\,\delta p}{A\rho'\delta x} = -\,\frac{1}{\rho'}\,.\,d_x p = \frac{E}{\rho'}\,.\,d^2_x y,$$

$$\therefore\ d_t^2 y = \frac{E}{\rho'}\,.\,d^2_x y.$$

Hence, by similar reasoning to that employed in Miller's Hydrostatics, p. 44, we find the velocity of sound to be $\sqrt{\frac{E}{\rho'}}$.

E has been supposed constant, but it is only so at a given uniform temperature; and as heat is given out by the body during the compression suffered by it at each pulsation, allowance must be made for the variation of temperature thus produced.

Ex. 18. To determine, by experiment, the velocity with which a ball is fired from a cannon.

Let the cannon be suspended in a horizontal position from a fixed point S, by means of a string or any other apparatus that will allow it to oscillate freely in a vertical plane; and observe the $\angle\,\theta$ through which it recoils when fired, and the number (n) of ocillations made in a given time t. Let G, O be the centres of gravity and oscillation of the cannon, M its mass; m, v the mass and velocity of the ball.

$$\text{Then } \frac{t}{n} = \pi\sqrt{\frac{\text{SO}}{g}},$$

and the moment of inertia of the cannon round S $=$ M . SG . SO (Art. 148), $\therefore$ it resists rotation as much as

$$\frac{\text{M . SG . SO}}{\text{SG}^2} = \text{M} . \frac{\text{SO}}{\text{SG}} \text{ placed at G, (Art. 146),}$$

and, by Art. 16, the momentum of the ball in one direction is equal to that of the cannon in the other,

$$\therefore \text{ the initial velocity of G} = \frac{m}{\text{M}} \cdot \frac{\text{SG}}{\text{SO}} \cdot v,$$

$$\therefore \text{ the initial velocity of O} = \frac{mv}{\text{M}}.$$

Now the cannon oscillates as if all its mass were collected at O, and $\therefore$ the height due to the velocity of O

$$= \frac{m^2 v^2}{\text{M}^2 . 2g}, \text{ which is } \therefore = \text{SO vers}\,\theta = 2\,\text{SO} . \sin^2\frac{\theta}{2};$$

$$\therefore v = \frac{\text{M}}{m} . 2 \sin\frac{\theta}{2} . \sqrt{\text{SO} . g},$$

$$= \frac{\text{M}}{m} . \frac{2tg}{n\pi} . \sin\frac{\theta}{2}.$$

The centres of gravity of the ball and cannon are supposed to be in the same horizontal line before the cannon is fired.

PRESSURE ON A FIXED AXIS.

171. When a body revolves about a fixed axis, each particle, being compelled to describe a circle, is endued with a centrifugal force tending directly from the axis; and this, with the impressed forces, would generally produce a motion of the axis itself, were it not kept fixed by the application of other forces to it, such as the reaction of the grooves in which it turns, &c. . . . if therefore we reckon amongst the impressed forces the reactions impressed upon the axis, we may consider the body as free, and the motion will be the same as before.

Angular motion is always supposed to take place from y to z, z to x, x to y, respectively, about the axes of x, y, z.

Ex. 1. A body acted on by no forces, revolves about a fixed axis; to determine the pressure on the axis.

Take the axis of z for that of rotation, $r =$ the distance of a particle m from it; let the plane xy pass through the centre of gravity of the body, $\omega =$ its $\angle^r$ velocity, xyz co-ordinates of m. The reaction of the axis may be resolved into a single force P acting at the origin in the plane xy, and making an $\angle\ \theta$ with the axis of x; and a couple $(\mathrm{S}, -\mathrm{S})$ acting at distances $(a, -a)$ from the origin, and making an $\angle\phi$ with the axis of x. Pritchard's Theory of Couples, prop. vii.

Then the impressed forces, in direction

$$\text{of } x = -\mathrm{P}\cos\theta - \mathrm{S}\cos\phi + \mathrm{S}\cos\phi = -\mathrm{P}\cos\theta,$$

$$\text{of } y = -\mathrm{P}\sin\theta - \mathrm{S}\sin\phi + \mathrm{S}\sin\phi = -\mathrm{P}\sin\theta,$$

and their moments about the axis of $x = 2\mathrm{S}a\sin\phi$, of $y = -2\mathrm{S}a\cos\phi$, of $z = 0$.

As the $\angle^r$ velocity is uniform, the only effective forces will be those that compel each particle, considered free, to describe a circle; they are therefore equal and opposite to the centrifugal forces; the effective moving force on m is $\therefore = -mr\omega^2$; and consequently the effective forces, in direction

$$\text{of } x = -\omega^2\Sigma mx, \text{ of } y = -\omega^2\Sigma my;$$

and their moments round the axis of $x = \omega^2\Sigma(myz)$, of $y = -\omega^2\Sigma(mxz)$, of $z = 0$.

Hence, by D'Alembert's principle, and the equations of equilibrium of a rigid body,

$$\mathrm{P}\cos\theta = \omega^2\Sigma mx, \ \mathrm{P}\sin\theta = \omega^2\Sigma my,$$

$$2\mathrm{S}a\sin\phi = \omega^2\Sigma(myz), \ 2\mathrm{S}a\cos\phi = \omega^2\Sigma(mxz).$$

If $\therefore$ $\dot{x}$ $\dot{y}$ be the co-ordinates of the centre of gravity,

$$\tan\theta = \frac{\Sigma my}{\Sigma mx} = \frac{\dot{y}}{\dot{x}}, \text{ and } \mathrm{P} = \omega^2\sqrt{\dot{x}^2 + \dot{y}^2}\,.\,\Sigma m,$$

$$\tan\phi = \frac{\Sigma(myz)}{\Sigma(mxz)}, \text{ and } 2\mathrm{S}a = \omega^2.\sqrt{(\Sigma mxz)^2 + (\Sigma myz)^2}.$$

Hence P is the same as if all the mass were collected at its centre of gravity. And if the axes of xyz be principal axes (Art. 159), the couple will vanish, and consequently there will be no torsion.

Cor. 1. A body will revolve permanently about a *free* axis, supposing it the axis of z, if $\Sigma(mxz) = 0$, $\Sigma(myz) = 0$. Consequently, the principal axes (Art. 159) are permanent axes.

Ex. 2. A body revolving about a fixed axis, is suddenly compelled to revolve about a new axis parallel to the former; to find the impulse upon the new axis at the very instant the body begins to revolve about it, and the new angular velocity.

Let m be the mass of the body, mk^2 its moment of inertia about an axis through its centre of gravity G (Fig. 58) $\parallel$ to the original axis $a\mathrm{A}a'$, and the new axis $b\mathrm{B}b'$; AGB is a plane through G $\perp$ to the axes; $\alpha = \angle$ BAG, $\beta = \angle$ ABG; ω, ω' the $\angle^r$ velocities about aa', bb' respectively.

The impulse may be supposed to take place at B; let X Y be the resolved parts of it $\parallel$ and $\perp$ to AB. Then, instead of having an axis at B, apply when the body is revolving about aa', the forces X Y in direction $\parallel$ and $\perp$ to BA, so that the point B may remain at rest.

Then the velocities due to the impulses, and communicated to the centre of gravity (Art. 137), and with it to every particle and $\therefore$ to B, are $\frac{X}{m}$, $\frac{Y}{m}$; also, the velocity of B about the centre of gravity (Art. 137), in consequence of the impulses $XY = \left(\frac{Y}{m} \cdot \cos\beta - \frac{X}{m} \cdot \sin\beta\right) . \frac{Ga^2}{k^2}$, which must be resolved in directions $\parallel$ and $\perp$ to BA; lastly, the point B had a previous velocity of ω . AB in direction $\perp$ to AB about aa'; and $\therefore$ since B remains at rest,

$$0 = -\frac{X}{m} + \left(\frac{Y}{m} . \cos\beta - \frac{X}{m} . \sin\beta\right) \cdot \frac{GB^2}{k^2} . \sin\beta,$$

$$\text{and } 0 = \frac{Y}{m} + \left(\frac{Y}{m} \cos\beta - \frac{X}{m} . \sin\beta\right) \cdot \frac{GB^2}{k^2} \cdot \cos\beta - \omega . Aa.$$

From which equations we obtain the impulse $\sqrt{X^2+Y^2}$,

$$= m . \omega . AB . \left\{\sin^2\beta + \left(\frac{k^2 \cos\beta}{k^2 + GB^2}\right)^2\right\}.$$

The velocity of the centre of gravity, after the impulses, $\parallel$ to AB ($= \omega'$. AG $\sin\beta$),

$$= \omega . AG . \sin\beta\left(1 - \frac{AG . AB}{k^2 + BG^2} \cdot \frac{\cos\alpha \sin\alpha}{\sin\beta}\right),$$

$$\therefore \omega' = \omega\left(1 - \frac{AG . AB}{k^2 + BG^2} \cdot \frac{\cos\alpha \sin\alpha}{\sin\beta}\right),$$

$$= \omega\left(1 - \frac{BG . AB}{k^2 + BG^2} \cdot \cos\alpha\right).$$

Ex. 3. A body acted on by given forces, revolves about a fixed axis; to find the pressure on the axis.

Let X Y Z be the forces impressed on the particle m; $\text{L} =$ the longitudinal pressure on the axis; the rest of the notation remaining as before. Hence the impressed forces in direction

of $x = \Sigma\text{X} - \text{P}\cos\theta$, of $y = \Sigma\text{Y} - \text{P}\sin\theta$, of $z = \Sigma\text{Z} + \text{L}$;
and their moments about the axis

$$\begin{aligned} &\text{of } x = \Sigma(\text{Z}y - \text{Y}z) + 2\text{S}a\sin\phi, \\ &\text{of } y = \Sigma(\text{X}z - \text{Z}x) - 2\text{S}a\cos\phi, \\ &\text{of } z = \Sigma(\text{Y}x - \text{X}y). \end{aligned}$$

The $\angle^r$ accelerating force $= d_t\omega$, and $\therefore$ the impressed moving forces on m, are $-mr\omega^2$ in direction of r, and $mrd_t\omega$ perpendicular to it; hence the effective forces on the body in the direction

$$\begin{aligned} &\text{of } x = -\omega^2\Sigma xm - d_t\omega\,.\,\Sigma ym, \\ &\text{of } y = -\omega^2\Sigma ym + d_t\omega\,.\,\Sigma xm, \qquad \text{of } z = 0, \end{aligned}$$

and their moments round the axis

$$\begin{aligned} &\text{of } x = \omega^2\Sigma(yzm) - d_t\omega\,.\,\Sigma(xzm), \\ &\text{of } y = -\omega^2\Sigma(xzm) - d_t\omega\,.\,\Sigma(yzm), \quad \text{of } z = d_t\omega\,.\,\Sigma(r^2m); \end{aligned}$$

and $\therefore$ by D'Alembert's principle, and the equations of equilibrium of a rigid body,

$$\Sigma\text{X} - \text{P}\cos\theta = -\omega^2\Sigma xm - d_t\omega\,.\,\Sigma ym \quad . \quad . \quad . \quad (1),$$

$$\Sigma\text{Y} - \text{P}\sin\theta = -\omega^2\Sigma ym + d_t\omega\,\Sigma xm \quad . \quad . \quad . \quad (2),$$

$$\Sigma\text{Z} + \text{T} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\Sigma(\text{Y}x - \text{X}y) = d_t\omega\,.\,\Sigma(r^2m) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\Sigma(\text{X}z - \text{Z}x) - 2\text{S}a\cos\phi = -\omega^2\Sigma(xzm) - d_t\omega\,.\,\Sigma(yzm) \quad . \quad (5),$$

$$\Sigma(\text{Z}y - \text{Z}z) + 2\text{S}a\sin\phi = \omega^2\Sigma(yzm) - d_t\omega\,.\,\Sigma(xzm) \quad . \quad . \quad (6).$$

From which six equations all the circumstances of the pressure may be determined.

Cor. 2. The angular accelerating force $d_t\omega$

$$= \frac{\Sigma(\text{Y}x - \text{X}y)}{\Sigma(r^2m)} = \frac{\text{moment of the forces}}{\text{moment of inertia}}\,. \quad \text{(Art. 144).}$$

As an application of these general formulæ, we may take the following problem.

A rigid uniform rod oscillates in a vertical plane about one end; to determine the pressure on the axis.

Let $2b$ = the length of the rod, M its mass, and suppose gravity to act in the direction of y, and the vertical plane to be that of xy. Then the six equations become

$$-\mathrm{P}\cos\theta = -\omega^2.\Sigma xm - d_t\omega.\Sigma ym = -(\omega^2\dot{x} + d_t\omega.\dot{y})\,\mathrm{M},$$

$$g.\mathrm{M} - \mathrm{P}\sin\theta = -\omega^2\,\Sigma ym + d_t\omega.\Sigma xm = -(\omega^2\dot{y} - d_t\omega.\dot{x})\,\mathrm{M},$$

$$g\dot{x}.\mathrm{M} = d_t\omega.\frac{4}{3}\mathrm{M}b^2,$$

$\therefore\ d_t\omega = \dfrac{3g\dot{x}}{4b^2} = \dfrac{3}{4}.\dfrac{g\cos\alpha}{b}$, α being the inclination of the rod to the axis of x; and

$$\tan\theta = \frac{1 + 3\sin^2\alpha + \dfrac{4b\omega^2}{g}.\sin\alpha}{\cos\alpha\left(3\sin\alpha + \dfrac{4b\omega^2}{g}\right)},$$

$$\mathrm{P} = \mathrm{M}g\sqrt{\left(\frac{b\omega^2}{g} + \frac{3}{4}\sin\alpha\right)^2\cos^2\alpha + \left(\frac{b\omega^2}{g}.\sin\alpha + \frac{3}{4}\sin^2\alpha + \frac{1}{4}\right)^2}.$$

Ex. 4. A quiescent body, acted on by no forces, and having a fixed axis, suddenly receives an impulse; to determine the initial pressure on the axis.

Let the impulse Y be communicated at the point $x'y'z'$, in a direction ∥ to the axis of y; then, using the same notation as before, and observing that $\omega = 0$, the six equations become

$$\mathrm{P}\cos\theta = d_t\omega.\Sigma ym,\ \mathrm{Y} - \mathrm{P}\sin\theta = d_t\omega.\Sigma xm,$$

$$\mathrm{Y}x' = d_t\omega.\Sigma(r^2m),$$

$$2\mathrm{S}a\cos\phi = d_t\omega.\Sigma(yzm),\ \mathrm{Y}z' - 2\mathrm{S}a\sin\phi = d_t\omega.\Sigma(xzm).$$

From which the pressure is to be determined.

Def. If $x'y'z'$ be so situated that there is no pressure on the axis, it is called the *centre of percussion*.

172. Ex. 5. To find the centre of percussion.

The only impressed force is Y at the point $x'y'z'$; and the only effective force on m, is $mr\,d_t\omega$ in a direction $\perp$ to r, which is equivalent to $-\,d_t\omega\,.\,\Sigma ym$, $d_t\omega\,.\,\Sigma xm$ in directions of x and y; hence, by D'Alembert's principle, and the conditions of equilibrium of a rigid body,

$$0 = d_t\omega\,.\,\Sigma ym,\quad \mathrm{Y} = d_t\omega\,.\,\Sigma xm,\quad \mathrm{Y}x' = d_t\omega\,.\,\Sigma\,(r^2m),$$

$$0 = d_t\omega\,.\,\Sigma(yzm),\quad \mathrm{Y}z' = d_t\omega\,.\,\Sigma\,(xzm).$$

Hence the conditions that there may be a centre of percussion are $\Sigma ym = 0$, $\Sigma\,(yzm) = 0$, and when these conditions are satisfied, its co-ordinates may be found from

$$x' = \frac{\Sigma\,(r^2m)}{\mathrm{Y}\,.\,\Sigma(xm)},\quad z' = \frac{\Sigma\,(xzm)}{\mathrm{Y}\,.\,\Sigma(xm)}.$$

y' is indeterminate.

ON THE MOTION OF A RIGID BODY ABOUT ITS CENTRE OF GRAVITY.

173. It was shewn in Art. 137, that the motion of any body whatever, acted on by given forces, is made up of the motion of the centre of gravity, and the motion of rotation about that point. The former is determined by supposing all the forces applied to the body, collected at its centre of gravity, and then using the equations of Art. 80; the latter, by supposing the centre of gravity a fixed point, as in the present section of the chapter.

174. Given $\omega_1\ \omega_2\ \omega_3$ the angular velocities of a body, estimated respectively about the axes of $x\ y\ z$ fixed in space, to find the linear velocities of any particle in the directions of the same axes.

Let xyz be the co-ordinates of the particle; then its linear velocity in consequence

of $\omega_1 = \omega_1 \sqrt{z^2 + y^2}$ which is equivalent to $0, -\omega_1 z, \omega_1 y$ in directions of the co-ordinates,

of $\omega_2 = \omega_2 \sqrt{x^2 + z^2}$ which is equivalent to $\omega_2 z, 0, -\omega_2 x$ in directions of the co-ordinates,

of $\omega_3 = \omega_3 \sqrt{x^2 + y^2}$ which is equivalent to $-\omega_3 y, \omega_3 x, 0$ in directions of the co-ordinates,

and consequently the velocity in directions of the co-ordinates are respectively

$$\omega_2 z - \omega_3 y, \quad \omega_3 x - \omega_1 z, \quad \omega_1 y - \omega_2 x.$$

175. Def. The axis about which the body is at any instant revolving, is called the ***axis of instantaneous rotation.***

For all points in this axis the linear velocities in directions of the co-ordinates are $= 0$.

$$\therefore 0 = \omega_2 z - \omega_3 y, \quad 0 = \omega_3 x - \omega_1 z, \quad 0 = \omega_1 y - \omega_2 x;$$

and $\therefore$ its equations are $\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3}$; and the cosines of the $\angle^s$ which it makes with the axes of $x\ y\ z$, are respectively

$$\frac{\omega_1}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}, \quad \frac{\omega_2}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}, \quad \frac{\omega_3}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}.$$

If P be a point in the axis of x, at distance 1 from the origin, its velocity $= \sqrt{\omega_2^2 + \omega_3^2}$, and its distance from the axis of instantaneous rotation $= \left(\frac{\omega_2^2 + \omega_3^2}{\omega_1^2 + \omega_2^2 + \omega_3^2}\right)^{\frac{1}{2}}$, and $\therefore$ the angular velocity of the body about this axis $=$ that of P about it $= \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$.

176. Cor. 1. Since $\omega_1 = (\angle^r \text{ vel. about OP}) . \cos xOP$, it follows that the $\angle^r$ velocity about one axis is resolved into $\angle^r$ velocity about any other, by multiplying by the cosine of their mutual inclination. If $\therefore$ ω be the $\angle^r$ velocity about the

axis of instantaneous rotation, and α, β, γ the $\angle^s$ which it makes with the co-ordinate axes, we have

$$\omega_1 = \omega \cos \alpha,\ \ \omega_2 = \omega \cos \beta,\ \ \omega_3 = \omega \cos \gamma.$$

177. Cor. 2. Hence supposing the axes of xyz to coincide with the principal axes of the body, at the time we are considering the motion, we have the *vis viva* of the body $= (\mathrm{A} \cos^2 \alpha + \mathrm{B} \cos^2 \beta + \mathrm{C} \cos^2 \gamma) . \omega^2$ (Arts. 140, Cor.; 162) $= \mathrm{A}\omega_1^2 + \mathrm{B}\omega_2^2 + \mathrm{C}\omega_3^2$,
which is constant if the body be acted on by no forces.

178. The sum of the products of each particle into the projection, on any plane, of twice the area described by its radius vector in a unit of time, is equal to the moment of inertia about an axis, through the origin perpendicular to that plane, multiplied by the angular velocity about the same axis.

For let the given plane be that of xy; $r =$ the distance of any particle m from the axis of z, $\theta =$ the $\angle$ which the projection of r on the plane xy makes with the axis of x.

Then the sum of the products $= \Sigma (mr^2 d_t\theta) = d_t\theta . \Sigma (mr^2)$ $= (\angle^r$ vel. about $z) .$ (mom. of inert. about z).

Hence, by Art. 130, when the particles are acted on by no other forces than their mutual influences, the product of the moment of inertia into the angular velocity, round any axis whatever, is constant.

179. Cor. Hence, on the same supposition as in Art. 177, the sum of the products for the plane $\perp$ to the axis of instantaneous rotation,

$$= (\mathrm{A} \cos^2 \alpha + \mathrm{B} \cos^2 \beta + \mathrm{C} \cos^2 \gamma)\, \omega,$$

$$= \frac{\mathrm{A}\omega_1^2 + \mathrm{B}\omega_2^2 + \mathrm{C}\omega_3^2}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}.$$

And on a plane the perpendicular to which makes any $\angle^s$ $\alpha' \beta' \gamma'$ with the principal axes, the sum
$= (A \cos^2 \alpha' + B \cos^2 \beta' + C \cos^2 \gamma')(\omega_1 \cos \alpha' + \omega_2 \cos \beta' + \omega_3 \cos \gamma')$.

180. Let xyz be the co-ordinates of a particle m referred to co-ordinate axes fixed in space, in such a position that at the time t when we are considering the motion they coincide with the principal axes of the body.

Then, by Art. 174,

$$d_t x = \omega_2 z - \omega_3 y, \quad d_t y = \omega_3 x - \omega_1 z, \quad d_t z = \omega_1 y - \omega_2 x.$$

$$\therefore d_t^2 x = z d_t \omega_2 - y d_t \omega_3 + \omega_2 d_t z - \omega_3 d_t y,$$
$$= z d_t \omega_2 - y d_t \omega_3 + \omega_2 (\omega_1 y - \omega_2 x) - \omega_3 (\omega_3 x - \omega_1 z).$$

Similarly,

$$d_t^2 y = x d_t \omega_3 - z d_t \omega_1 + \omega_3 (\omega_2 z - \omega_3 y) - \omega_1 (\omega_1 y - \omega_2 x),$$
$$d_t^2 z = y d_t \omega_1 - x d_t \omega_2 + \omega_1 (\omega_3 x - \omega_1 z) - \omega_2 (\omega_2 z - \omega_3 y).$$

Hence the moment of the effective forces about the axis of z

$$= \Sigma m \, (x d_t^2 y - y d_t^2 x),$$
$$= d_t \omega_3 \, . \, \Sigma \, (x^2 + y^2) \, m + \omega_1 \omega_2 \, \Sigma \, (x^2 - y^2) \, m,$$
$$= d_t \omega_3 \, . \, \Sigma \, (x^2 + y^2) \, m + \omega_1 \omega_2 \, \Sigma \{ x^2 + z^2 - (y^2 + z^2) \} \, m,$$
$$= C \, d_t \omega_3 + (B - A) \, \omega_1 \omega_2.$$

Hence if L, M, N be the moments of the impressed forces about the axes of x, y, z, we shall have, by D'Alembert's principle, and the equations of equilibrium of a rigid body,

$$\left.\begin{aligned} N &= C \, d_t \omega_3 + (B - A) \, \omega_2 \omega_1 \\ \text{similarly, } M &= B \, d_t \omega_2 + (A - C) \, \omega_1 \omega_3 \\ L &= A \, d_t \omega_1 + (C - B) \, \omega_3 \omega_2 \end{aligned}\right\} \ldots (A).$$

By the integration of these equations we shall obtain the angular velocities of the body about the principal axes. Art. 176 will then enable us to find the angular velocities about any axes whatever.

181. We shall now shew how from these angular velocities we are able to determine the position of the body.

Let G the centre of gravity and origin of co-ordinates be in the centre of a spherical surface whose radius is 1, and let this surface cut the principal axes in the points xyz (Fig. 59), and any axes fixed in space and having the origin G in $x'\,y'\,z'$. Join these points by arcs of great circles, as in the Figure, and let Nn be the node line of the planes of xy, $x'y'$; $\phi = \mathrm{N}x = pzy$, $\psi = \mathrm{N}x' = pz'y'$, $\theta = x\mathrm{N}x' = zz'$. Then resolving (by Art. 176) the angular velocities $d_t\phi$, $d_t\psi$, $d_t\theta$ (which take place round the axes oz, oz', oN) into angular velocities about the axes ox, oy, oz, we obtain

$$\left.\begin{aligned}
\omega_1 &= -\,d_t\psi\,.\cos z'x - d_t\theta\,.\cos \mathrm{N}x + d_t\phi\,.\cos zx \\
&= d_t\psi\,.\sin\theta\sin\phi - d_t\theta\,.\cos\phi \;\ldots\ldots\ldots \\
\omega_2 &= -\,d_t\psi\,.\cos z'y - d_t\theta\,.\cos \mathrm{N}y + d_t\phi\,.\cos zy \\
&= d_t\psi\,.\sin\theta\,.\cos\phi + d_t\theta\,.\sin\phi \;\ldots\ldots\ldots \\
\omega_3 &= -\,d_t\psi\,.\cos z'z - d_t\theta\,.\cos \mathrm{N}z + d_t\phi\,.\cos zz \\
&= d_t\phi - d_t\psi\,.\cos\theta \;\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}\ldots(\mathrm{B}),$$

from which equations, when $\omega_1\,\omega_2\,\omega_3$ are known, we shall, by integration, be able to determine ϕ, ψ, θ.

182. We may however, without the aid of Art. 180, find certain relations between the angular velocities $\omega_1\,\omega_2\,\omega_3$.

Using the notation of Art. 181, let L′ M′ N′ be the moments of the impressed forces about the axes of $x'\,y'\,z'$, then

$$\Sigma m\,(y'd_t^2z' - z'd_t^2y') = \mathrm{L}', \quad \Sigma m\,(z'd_t^2x' - x'd_t^2z') = \mathrm{M}',$$
$$\Sigma m\,(x'd_t^2y' - y'd_t^2x') = \mathrm{N}';$$

and ∴ by integration

$\mathrm{C}_1 + \int_t \mathrm{L}' = \Sigma m\,(y'd_tz' - z'd_ty')$ = twice the sum of the products of each particle into the projections, on the plane of $y'z'$, of the areas described in a unit of time.

But, by Art. 178, these sums for the projections on the planes yz, xz, xy, are $\mathrm{A}\omega_1$, $\mathrm{B}\omega_2$, $\mathrm{C}\omega_3$ respectively. If therefore $a\,b\,c$ be the cosines of the inclinations of the axis of x' to the axes of $x\,y\,z$; $a'b'c'$, $a''b''c''$, similar quantities for the axes of $y'\,z'$; by Hymer's Anal. Geom. art. 17, we shall have

$$\left.\begin{aligned}
&\mathrm{C}_1 + \textstyle\int_t \mathrm{L}' = \mathrm{A}a\omega_1 + \mathrm{B}b\omega_2 + \mathrm{C}c\omega_3 \\
\text{similarly, } &\mathrm{C}_2 + \textstyle\int_t \mathrm{M}' = \mathrm{A}a'\omega_1 + \mathrm{B}b'\omega_2 + \mathrm{C}c'\omega_3 \\
&\mathrm{C}_3 + \textstyle\int_t \mathrm{N}' = \mathrm{A}a''\omega_1 + \mathrm{B}b''\omega_2 + \mathrm{C}c''\omega_3
\end{aligned}\right\}\ldots\ (\mathrm{C}).$$

183. By referring to Fig. 59, we find

$$\begin{aligned}
a &= \cos x'x = \cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta,\\
b &= \cos x'y = -\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta,\\
c &= \cos x'z = \sin\theta\sin\psi,\\
a' &= \cos y'x = -\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta,\\
b' &= \cos y'y = \sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta,\\
c' &= \cos y'z = \sin\theta\cos\psi,\\
a'' &= \cos z'x = -\sin\theta\sin\phi,\\
b'' &= \cos z'y = -\sin\theta\cos\phi,\\
c'' &= \cos z'z = \cos\theta.
\end{aligned}$$

184. In the particular case, when the body is acted on by no forces; to determine the motion.

Here L' M' N' are each $= 0$,

$$\begin{aligned}
\therefore\ & A a\,\omega_1 + B b\,\omega_2 + C c\,\omega_3 = C_1,\\
& A a'\omega_1 + B b'\omega_2 + C c'\omega_3 = C_2,\\
& A a''\omega_1 + B b''\omega_2 + C c''\omega_3 = C_3;\\
\therefore\ & A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = C_1^2 + C_2^2 + C_3^2 = k^2;
\end{aligned}$$

and, by Art. 177,

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = \text{constant} = h^2.$$

185. Cor. 1. If a line be drawn making $\angle^s$ with x' y' z' whose cosines are $\frac{C_1}{k}$, $\frac{C_2}{k}$, $\frac{C_3}{k}$, it will remain fixed during the whole motion of the body. It is the plane perpendicular to this line which, in Art. 132, is called the principal plane of moments.

186. Cor. 2. To find the inclination of this line to the principal axes, and the angular velocity about it.

The angular velocity ω takes place about the axis of instantaneous rotation, which is inclined to the principal axes at $\angle^s$

whose cosines are $\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}$ (Art. 175); and the cosines of the $\angle^s$ which the line in question makes with the same axes are

$$\left(\frac{C_1}{k}.a+\frac{C_2}{k}.a'+\frac{C_3}{k}.a''\right), \quad \left(\frac{C_1}{k}.b+\frac{C_2}{k}.b'+\frac{C_3}{k}.b''\right),$$
$$\left(\frac{C_1}{k}.c+\frac{C_2}{k}.c'+\frac{C_3}{k}.c''\right),$$

or, (by substitution for C_1, C_2, C_3) $\frac{A\omega_1}{k}, \frac{A\omega_2}{k}, \frac{A\omega_3}{k}$, and the angular velocity about it $= \frac{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}{k} = \frac{h^2}{k}$, which is $\therefore$ uniform.

Since this line is fixed, and its position has been found, it will be convenient to take it for the axis of z'.

$$\therefore \frac{A\omega_1}{k} = a'' = -\sin\theta\sin\phi, \quad \frac{B\omega_2}{k} = b'' = -\sin\theta . \cos\phi,$$
$$\frac{C\omega_3}{k} = c'' = \cos\theta,$$

by means of which, and equations (A), the motion may be determined.

Ex. 1. Let A B C be all equal.

Then equations (A) shew that ω_1, ω_2, ω_3 are constant quantities; and consequently $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$ are also constant.

Hence the axis of instantaneous rotation is fixed both in the body and in space, and the body revolves about it with a uniform velocity.

Ex. 2. Let A, B be equal.

Equations (A) give
$$d_t\omega_3 = 0,$$
$$A d_t\omega_1 - (A - C)\,\omega_2\,\omega_3 = 0,$$
$$A d_t\omega_2 + (A - C)\,\omega_1\,\omega_3 = 0,$$

Hence $\omega_3 =$ constant, from the first; and by eliminating $(A - C)$ from the second and third,

$$\omega_1 d_t\omega_1 + \omega_2 d_t\omega_2 = 0, \;\therefore\; \omega_1^2 + \omega_2^2 = \text{constant}.$$

and $\therefore$ the volocity about the axis of instantaneous rotation $\left(= \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}\right)$ is constant; and since $\frac{\omega_3}{\omega}$ and $\cos\theta$ $\left(= \frac{C\omega_3}{k}\right)$ and $\therefore$ θ are constant, it follows that that axis does itself revolve uniformly about the axis of z, describing a conical surface about it; and that this axis of z at the same time describes a conical surface about the axis of z', which is fixed in space.

Again, by eliminating ω_2 from the two last of the three equations above written, we obtain

$$d_t^2\omega_1 + \left(1 - \frac{C}{A}\right)^2 \omega_3^2 . \omega_1 = 0.$$

The integral of which is

$$\omega_1 = K \sin(nt + e), \text{ where } n = \left(1 - \frac{C}{A}\right) . \omega_3.$$

$$\therefore \omega_2 = \frac{d_t\omega_1}{n} = K \cos(nt + e),$$

$$\therefore \omega_1^2 + \omega_2^2 = K^2.$$

Again,

$$\tan\phi = \frac{\omega_1}{\omega_2} = \tan(nt + e),$$

$$\therefore \phi = nt + e, \ \therefore d_t\phi = \omega_3 . \left(1 - \frac{C}{A}\right),$$

$$\text{and } d_t\psi = \frac{d_t\phi - \omega_3}{\cos\theta} \text{ (Art. 181) } = \frac{-k}{A}.$$

Hence the axis of z revolves uniformly about the axis of z', and the motion may be thus exhibited mechanically.

Let PQ, PR (Fig. 60) be wheels whose axes Gz', Gz are inclined at an $\angle\ \theta$ to each other, and their radii in the proportion of $\omega_3\left(1 - \frac{C}{A}\right) : \frac{k}{A}$. Then if the axis G$z$ be made to revolve uniformly about Gz' with an $\angle^r$ velocity $\frac{-k}{A}$, the wheel PR rolling upon PQ, the motion of the body will be truly represented.

Ex. 3. A, B, C being all unequal; to determine the motion, supposing that the axis of z nearly coincides with the axis of instantaneous rotation.

Since the sine of their inclination $= \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega}$ (Art. 175), is very small, $\therefore$ ω_1, ω_2 are each very small; and $\therefore$ neglecting their product, equations (A) become

$$\begin{aligned} 0 &= C d_t \omega_3 \text{ very nearly,} \\ 0 &= B d_t \omega_2 + (A - C)\, \omega_1\, \omega_3, \\ 0 &= A d_t \omega_1 + (C - B)\, \omega_3\, \omega_2. \end{aligned}$$

$\therefore$ ω_3 is very nearly constant, and consequently the velocity about the axis of z is very nearly uniform. Since $\cos\theta = \frac{C\omega_3}{k}$ (Art 186), $\therefore$ θ, or the inclination of the axis of z', is nearly constant; and from the last but one of the equations in Art. 184, $k = C\omega_3$ nearly, and $\therefore$ θ is very small; the second and third of the above equations, being united, give

$$\omega_1 = \sqrt{\left(\frac{k^2 - Ch^2}{AC - A^2}\right)} \cdot \sin(nt + e),$$

$$\omega_2 = \sqrt{\left(\frac{B(C - A)(k^2 - Ch^2)}{A(C - B)(AC - A^2)}\right)} \cdot \cos(nt + e),$$

where $n = \omega_3 \cdot \sqrt{\left\{\left(\frac{C}{A} - 1\right)\left(\frac{C}{B} - 1\right)\right\}}$; also the last of equations (B) in Art. 181, becomes, by integration, &c.

$$\psi = \phi - \omega_3 t + e';$$

and, from Art. 186,

$$\tan\phi = \frac{A\omega_1}{B\omega_2} = \frac{A}{B} \cdot \sqrt{\left(\frac{A(C - B)}{B(C - A)}\right)} \tan(nt + e).$$

Cor. 1. When C − A and C − B have the same sign, the above expressions are all possible, and the body will go on perpetually revolving about an axis very nearly coinciding with that principal axis about which the moment of inertia is

C; but if C be the mean moment of inertia, the above expressions will become exponential quantities, which will continually increase with the time; and $\therefore$ a body cannot go on perpetually revolving about an axis nearly coinciding with the axis of mean moment, but the axis of rotation will recede farther and farther from it.

Cor. 2. If at the commencement of the motion, the body revolve about a principal axis, we have $\omega_1 = 0$, $\omega_2 = 0$ when $t = 0$, and $\therefore k^2 - ch^2 = 0$, and $\therefore$ always $\omega_1 = 0$, and $\omega_2 = 0$, and consequently it will revolve perpetually about that axis; on which account the principal axes are called *natural axes of rotation*. And conversely, if ever $\omega_1 = 0$, and $\omega_2 = 0$, at the same instant, the body must of necessity be revolving about a principal axis. Hence, if at the commencement of the motion the axis of rotation did not coincide with a principal axis, it never will coincide with one.

That the axis of instantaneous rotation may always preserve the same position with respect to the body, it is necessary that ω_1, ω_2, ω_3 be each constant, and $\therefore d_t\omega_1 = 0$, $d_t\omega_2 = 0$, $d_t\omega_3 = 0$, which reduce the three equations (A) to

$$0 = (B-A)\,\omega_1\,\omega_2,\quad 0 = (A-C)\,\omega_1\,\omega_3,\quad 0 = (C-B)\,\omega_2\,\omega_3;$$

to satisfy which it is necessary to suppose two of the three quantities ω_1, ω_2, ω_3 evanescent, in which case the body will revolve about a principal axis. Hence none but a principal axis can be a perpetual axis of rotation.

The rotation is said to be *stable* about the axes of greatest and least moments of inertia, and *unstable* about the other.

Ex. 4. A, B, C being all unequal, and the axis of instantaneous rotation any where situated; to determine the motion.

From the two last of the equations in Art. 184, we obtain

$$\omega_1^2 = \frac{k^2 - Bh^2 + C\,(B - C)\,\omega_3^2}{A\,(A - B)},$$

$$\omega_2^2 = \frac{k^2 - Ah^2 + C\,(A - C)\,\omega_3^2}{B\,(B - A)},$$

and eliminating ω_1, ω_2 from the first of equations (A), $d_{\omega_3} t =$

$$\frac{C\,.\,\sqrt{AB}}{\sqrt{\{k^2 - Bh^2 + (B - C)\,C\omega_3^2\}\,.\,\{-k^2 + Ah^2 + (C - A)\,C\omega_3^2\}}},$$

from which t and ω_3 are known, by integration, in terms of each other, and the above two equations will determine ω_1, ω_2 in terms of t, and the equations of Art. 183

$$\frac{A\omega_1}{k} = -\sin\theta\sin\phi,\quad \frac{B\omega_2}{k} = -\sin\theta\cos\phi,\quad \frac{C\omega_3}{k} = \cos\theta,$$

will determine the $\angle^s$ ϕ, θ; and, by eliminating $d_t\theta$ from the two first of equations (B),

$$d_t\psi = \frac{\omega_1\sin\theta\sin\phi + \omega_2\sin\theta\cos\phi}{1 - \cos^2\theta} = -\frac{k\,(A\omega_1^2 + B\omega_2^2)}{k^2 - C^2\omega_3^2},$$

$$= -\frac{k\,(h^2 - C\omega_3^2)}{k^2 - C^2\omega_3^2},$$

$$\therefore d_{\omega_3}\psi = \frac{k\,(C\omega_3^2 - h^2)\,C\sqrt{AB}}{(k^2 - C^2\omega_3^2)\sqrt{\{k^2 - Bh^2 + (B - C)\,C\omega_3^2\}\,.\,\{-k^2 + Ah^2 + (C - A)\,C\omega_3^2\}}},$$

which determines the value of ψ.

Hence ω_1, ω_2, ω_3, θ, ϕ, ψ, are all determined in terms of t, and $\therefore$ the position of the body, at any instant, is completely known.

Cor. Eliminating ω_3 from the two last of the equations in Art. 184, we obtain

$$(A - C)\,.\,A\omega_1^2 + (B - C)\,.\,B\omega_2^2 = k^2 - Ch^2.$$

If $\therefore$ the axis of z be the principal axis about which the moment of inertia is a maximum or a minimum, ω_1 and ω_2 cannot increase beyond certain limits; but will always be respectively less than

$$\left(\frac{k^2 - Ch^2}{A^2 - AC}\right)^{\frac{1}{2}} \text{ and } \left(\frac{k^2 - Ch^2}{B^2 - BC}\right)^{\frac{1}{2}};$$

and on the same supposition, if at any time ω_1, ω_2 happen to be very small, they will always be small.

CHAPTER X.

ON THE MOTION OF A PARTICLE OF MATTER IN A RESISTING MEDIUM.

187. In what has hitherto been done, the motion of a body has always been supposed to take place in a vacuum; if it move in a medium, it must displace the successive particles with which it comes in contact, and as they, in common with all matter, are possessed of inertia, they will, by Art. 23, exert a resisting force upon the body, equal and opposite to that which the body exerts upon them. Hence the resistance of a medium is of the nature of a pressure, and takes place in a direction opposite to the motion; and if we reckon this pressure amongst the forces impressed upon the body, we may consider it as being in a vacuum, and proceed as in the preceding chapters. It is evident from the above explanation, that the resistance of a medium depends partly on the density of the medium, and partly on the velocity of the body. It is found, by experiment, that the resistance to a given body $\propto$ (density) . (velocity)^2 nearly, when the velocity is not very great or very small. We may therefore, in general, represent the resistance by $Mkgv^2$; where M is the mass of the body; k a quantity depending on the form of the body, and the density and nature of the medium, which must be determined by experiment; and g = the relative gravity of the body = $\left(1 - \dfrac{\text{specific gravity of the medium}}{\text{specific gravity of the body}}\right)$. (force of gravity). In this chapter g is always used as here defined. Under the

same limitations as before, the resistance to a plane surface is equal to the weight of a uniform column of the fluid whose base is the plane surface, and altitude the space through which a body must fall by the action of gravity to acquire its velocity. Notwithstanding the resistance in general $\propto$ (velocity)^2, it will sometimes, for the sake of examples, be supposed to vary according to other laws.

188. A particle acted on by no force is projected in a uniform medium, the resistance of which varies as the velocity; to determine the motion.

Since the resistance acts in the direction opposite to the motion, the body will evidently move in a straight line; let s be the space described in the time t; and suppose the resistance $= Mkgv$.

$$\therefore\ vd_sv = -\,kgv,\ \therefore\ v = kg\,(a - s);$$

a being the space through which the particle must move before it loses its whole velocity $= \dfrac{\text{velocity of projection}}{kg}$.

$$\text{Again, } d_tv = -\,kgv,$$

$$\therefore\ v = e^{-kgt}.\ (\text{velocity of projection}).$$

189. A body of given elasticity descends from rest through a given altitude, acted on by the force of gravity in a uniform medium, of which the resistance $= Mkgv^2$, and impinging on a horizontal plane, rises and falls alternately; to determine the whole space described. See Ex. 3, p. 26.

Let a be the given altitude, λ the elasticity of the body; a_x the space described in the x^{th} descent; s a portion of that space descended through in the time t, v the velocity.

$$\text{Then } vd_sv = g - gkv^2,\ \therefore\ v^2 = \frac{1}{k}\cdot(1 - e^{-2gks}),$$

and $\therefore$ the (vel.)^2 acquired in the x^{th} descent $= \dfrac{1}{k}\cdot(1 - e^{-2gka_x})$.

and $\therefore$ at the beginning of the next ascent,

$$(\text{vel.})^2 = \frac{\lambda^2}{k} \cdot (1 - e^{-2gka_x}) = v'^2.$$

Let now s be the space ascended in the time t, when the velocity is v,

$$\therefore v d_s v = -g - gkv^2, \therefore v^2 = \left(v'^2 + \frac{1}{k}\right) . e^{-2gks} - \frac{1}{k}.$$

Now when $v = 0$, $s = a_{x+1}$,

$$\therefore e^{2gka_{x+1}} - 1 = kv'^2 = \lambda^2 - \lambda^2 e^{-2gka_x},$$

For e^{2gka_x} write $\frac{u_{x+1}}{u_x}$,

$$\therefore u_{x+2} - (1 + \lambda^2)\, u_{x+1} + \lambda^2 u_x = 0.$$

$$\therefore u_x = \text{C}\lambda^{2x} + \text{CB}, \text{ by } \textit{Finite Differences}.$$

$$\therefore e^{2gka_x} = \frac{\lambda^{2x+2} + \text{B}}{\lambda^{2x} + \text{B}}.$$

Hence we know the value of B from the first descent, which gives

$$e^{2gka} = \frac{\lambda^4 + \text{B}}{\lambda^2 + \text{B}}.$$

Again, by continual multiplication, we have

$$e^{2gk(a_2 + a_3 + a_4 + \cdots\cdots + a_x)} = \frac{\lambda^6 + \text{B}}{\lambda^4 + \text{B}} \cdot \frac{\lambda^8 + \text{B}}{\lambda^6 + \text{B}} \cdot \frac{\lambda^{10} + \text{B}}{\lambda^8 + \text{B}} \cdots\cdots \frac{\lambda^{2x+2} + \text{B}}{\lambda^{2x} + \text{B}}.$$

$$= \frac{\lambda^{2x+2} + \text{B}}{\lambda^4 + \text{B}}.$$

$$\therefore e^{2gk(a_2 + a_3 + a_4 + \cdots\cdots)} = \frac{\text{B}}{\lambda^4 + \text{B}}, \text{ when } x \text{ is infinite};$$

$$= \frac{1 - \lambda^2 . e^{-2gka}}{1 - \lambda^2}.$$

$\therefore$ the whole space described

$$= a + 2a_2 + 2a_3 + 2a_4 + \&\text{c}.$$

$$= a + \frac{1}{kg} \cdot \log_e \left(\frac{1 - \lambda^2 . e^{-2gka}}{1 - \lambda^2}\right).$$

See Cambridge Problems for 1824.

190. If chords be drawn from the extremity of the vertical diameter of a circle, the velocities acquired by a particle descending down them from rest in a medium, the resistance of which $\propto$ velocity, will vary as their lengths, and the times of descent will be equal. See Art. 60.

Let AB (Fig. 5) be a vertical diameter, CB a chord, CD $\perp$ AB; $s =$ any space descended from C down the chord, $v =$ the velocity. Then,

$$d_t^2 s = g \cdot \frac{\mathrm{BD}}{\mathrm{CB}} - kg \cdot v = g \cdot \frac{\mathrm{BD}}{\mathrm{CB}} - kg \cdot d_t s,$$

$$\therefore\ d_t s + kgs = gt \cdot \frac{\mathrm{BD}}{\mathrm{CB}},$$

$$\therefore\ s = \frac{\mathrm{BD}}{\mathrm{CB}} \cdot \frac{t}{k} - \frac{\mathrm{BD}}{\mathrm{CB}} \cdot \frac{1 - e^{-kgt}}{k^2 g},$$

$$\therefore\ \mathrm{AB} = \frac{\mathrm{CB}^2}{\mathrm{BD}} = \frac{t}{k} - \frac{1 - e^{-kgt}}{k^2 g}.$$

Hence t being constant will be the same for all chords.

The velocity acquired down CB $\propto d_t s \propto \frac{\mathrm{BD}}{\mathrm{CB}} \propto \frac{\mathrm{CB.BD}}{\mathrm{CB}^2} \propto$ CB.

191. A body acted on by gravity, is projected from a given point with a given velocity, and in a given direction in a uniform medium, the resistance of which varies as the velocity; to determine the motion.

Making the same construction as in Art. 69, and using the same notation, except that $g =$ the relative gravity of the body, we have

$$d_t^2 x = -kg d_t x, \quad d_t^2 y = -g - kg d_t y,$$

M$kg\, d_t s$ being the resistance.

From the first equation, $x = \mathrm{B}e^{-kgt} + \mathrm{B}'$, and $\therefore\ d_t x = -\mathrm{B}kge^{-kgt}$.

And since the second equation may be put under the form of the first, thus

$$d_t^2\left(\frac{t}{k} + y\right) = -kg\, d_t\left(\frac{t}{k} + y\right),$$

we have

$$\frac{t}{k} + y = Ce^{-kgt} + C', \text{ and } \therefore d_t y = -Ckge^{-kgt} - \frac{1}{k}.$$

Now when $t = 0, x = 0, y = 0, d_t x = v\cos\theta, d_t y = v\sin\theta$,

$$\therefore B' = -B = \frac{v}{kg}\cdot\cos\theta,\ C_1 = -C = \frac{v}{kg}\cdot\sin\theta + \frac{1}{k^2 g},$$

$$\therefore x = \frac{v\cos\theta}{kg}\cdot(1 - e^{-kgt}),$$

$$\therefore -\frac{t}{k} = \frac{1}{k^2 g}\cdot\log_e\left(1 - \frac{kgx}{v\cos\theta}\right);$$

also, $y = \left(\frac{v\sin\theta}{kg} + \frac{1}{k^2 g}\right)\cdot(1 - e^{-kgt}) - \frac{t}{k}$,

$$= \left(\tan\theta + \frac{1}{kv\cos\theta}\right)\cdot x + \frac{1}{k^2 g}\cdot\log_e\left(1 - \frac{kgx}{v\cos\theta}\right),$$

which determines the path of the projectile.

Cor. 1. By expanding the expression for y, we have

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} - \frac{kg^2x^3}{3v^3\cos^3\theta} - \&c.$$

which coincides with the expression in Art. 69, when $k = 0$; the terms containing k shew the effect of the resistance in making the projectile deviate from the parabola it would describe in a vacuum.

Cor. 2. The greatest value of x is $\frac{v\cos\theta}{kg}$; for if x be increased beyond this limit, the value of y becomes imaginary; and since

$$d_x y = \tan\theta + \frac{\sec\theta}{kv} - \frac{1}{k}\cdot\frac{1}{v\cos\theta - kgx},$$

the tangent to the curve is then $\perp$ to the axis of x; hence making AE (Fig. 61) equal to $\frac{v\cos\theta}{kg}$, and drawing EZ $\perp$ to the horizon, EZ will be an asymptote to the curve.

x may be increased in a negative direction without limit, but there will be no asymptote to that branch of the curve; the inclination when x and y are both infinite and negative $=$ $\tan^{-1}\left(\tan\theta + \frac{\sec\theta}{kv}\right)$.

COR. 3. The range AB on a horizontal plane may be found by making $y = 0$, in the equation of the curve; and by making $d_x y = 0$, we have the greatest altitude

$$= \frac{\sin\theta}{kg} - \frac{1}{k^2 g}\cdot \log_e (1 + kv \sin\theta);$$

and the corresponding value of x

$$= \frac{v^2}{g}\cdot\frac{\sin\theta\cos\theta}{1 + kv\sin\theta}.$$

COR. 4. Let the tangent at P be produced to meet the asymptote in T; then if $s = \text{AP}$,

$$\text{PT} = (\text{AE} - x).d_x s = \left(\frac{v\cos\theta}{kg} - x\right).d_t s.d_x t = \frac{d_t s}{kg};$$

$\therefore$ the velocity $\propto$ the length of the tangent produced to meet the asymptote.

The velocity approaches continually to $\frac{1}{k}$ as its limit, and $\therefore$ the ultimate value of PT is $\frac{1}{k^2 g}$.

192. In the last problem, supposing the resistance to vary as the square of the velocity; to determine the motion.

In this case

$$d_t^2 x = -kg d_t s.d_t x, \quad d_t^2 y = -g - kg d_t s d_t y.$$

Making x the independent variable, and reducing, these equations become

$$\frac{d_x^2 t}{d_x t} = kg d_x s, \quad d_x^2 y = -g d_x t^2,$$

the former of these gives at once, by integration, &c.

$$d_x t = \mathrm{C}e^{kgs} = \frac{e^{kgs}}{v\cos\theta},$$

and the latter, by writing p for $d_x y$, becomes

$$d_x p = -g d_x t^2 = \frac{-ge^{2kgs}}{v^2\cos^2\theta}, \quad \ldots\ldots\ldots \quad (1)$$

$$\therefore d_x p \,.\, \sqrt{1+p^2} = -\frac{ge^{2kgs}d_x s}{v^2\cos^2\theta}; \quad \ldots\ldots\ldots\ldots \quad (2)$$

and therefore, by integrating, we obtain

$$\mathrm{C}' + p\sqrt{1+p^2} + \log_e(p+\sqrt{1+p^2}) = -\frac{e^{2kgs}-1}{kv^2\cos^2\theta} \ldots (3)$$

$$\therefore \frac{e^{2kgs}-1}{2kg} = -\frac{v^2}{2g}\cdot\cos^2\theta\{\mathrm{C}'+p\sqrt{1+p^2}+\log_e(p+\sqrt{1+p^2})\},$$

= arc S of the parabola described in a vacuum by the action of the force g (Art. 74),

$$\therefore 2kgs = \log_e(1+2kg\mathrm{S}), \quad \ldots\ldots \quad (4).$$

The length of the parabolic arc S is such, that the tangent at its extremity is || to the tangent at the extremity of the arc s.

The equation of the curve will be obtained by eliminating p between $p = d_x y$ and (3), and then integrating. No method, however, of doing this in finite terms has yet been discovered, although the problem, which is no other than to determine the path of a cannon ball when fired into the air, has exercised the abilities of the most celebrated writers on Mechanics. Many of its principal properties may be exhibited by means of equation (4).

Cor. 1. s and S increase together, but S much faster than s, and they are infinite together; but when S is infinite, it is well known that $p = \infty$; and by differentiating equation (1), and substituting in (2), we have

$d_x^2 p = 2kg d_x p\cdot\sqrt{1+p^2} = -2kgp d_x p$ nearly, when p is very large;

$$\therefore d_x p = -kgp^2:$$

no constant being added, because it vanishes compared with p^2.

Dividing the last equation by p^2, and integrating, we obtain

$$\frac{1}{p} = \mathrm{C}'' - kgx.$$

Since this expression is nearly true, when p is a very large quantity, C″ is of necessity finite, and ∴ when p is infinite

$$x = \frac{\mathrm{C}''}{kg},$$

hence if AE (Fig. 61) be made $= \frac{\mathrm{C}''}{kg}$, EZ, which is ⊥ AE, will be an asymptote to the descending branch.

Cor. 2. In the ascending branch, when s and S are negative, equation (4) becomes

$$-2kgs = \log_e (1 - 2kg\mathrm{S}),$$

from which we see that s is infinite when

$$\mathrm{S} = \frac{1}{2kg};$$

if therefore we take an arc of the parabola $= \frac{1}{2kg}$, a tangent at its extremity will be ∥ to the ascending branch at an infinite distance, which shews that the ascending and descending branches are dissimilar, the body rising obliquely and ultimately descending vertically.

From (3) it is evident, when s has a very large negative value, p = constant nearly,

$$\therefore d_x y = \mathrm{A} \text{ nearly},$$

$$\therefore y = \mathrm{A}x + \mathrm{A}' \text{ nearly};$$

which equation being true when x and y are large finite quantities, A′ must be finite, and ∴ the ascending branch has an asymptote whose equation is

$$y = \mathrm{A}x + \mathrm{A}'.$$

Cor. 3. In the practice of gunnery p is always very small, and in this case

$$\frac{-g e^{2kgs} d_x s}{v^2 \cos^2\theta} = d_x p \sqrt{1+p^2} = d_x p + \tfrac{1}{2} p^2 d_x p, \text{ nearly;}$$

$$\therefore \mathrm{C} - \frac{e^{2kgs}-1}{2kv^2\cos^2\theta} = p + \frac{p^3}{6} = p, \ \therefore \mathrm{C} = \tan\theta;$$

$$\therefore e^{2kgs} = 1 + 2kv^2\cos^2\theta\,(\tan\theta - p).$$

Substituting this in equation (1),

$$1 = \frac{-v^2\cos^2\theta\, d_x p}{g + 2kgv^2\cos^2\theta\,(\tan\theta - p)},$$

$$\therefore d_x y = p = \tan\theta - \frac{e^{2kgx}-1}{2kv^2\cos^2\theta},$$

$$\therefore y = x\left(\tan\theta + \frac{\sec^2\theta}{2kv^2}\right) - \frac{e^{2kgx}-1}{4k^2gv^2\cos^2\theta}.$$

193. **A particle moves in a resisting medium acted on by a given force in parallel lines; to find the resistance that a given curve may be described.**

Let Y be the force acting on the particle at the point xy, where the resistance is R; s = the arc described.

Then $\quad d_t^2 x = -\mathrm{R} d_t x, \ d_t^2 y = -\mathrm{Y} - \mathrm{R} d_t y;$

changing the independent variable, these equations become

$$\frac{d_x^2 t}{d_x t^3} = \frac{\mathrm{R}}{d_x s},$$

$$\text{and } \frac{d_x^2 y}{d_x t^2} - \frac{d_x y\, d_x^2 t}{d_x t^3} = -\mathrm{Y} - \mathrm{R}\,.\,\frac{d_x y}{d_x s} = -\mathrm{Y} - \frac{d_x y\, d_x^2 t}{d_x t^3}.$$

$$\therefore \frac{1}{d_x t^2} = -\frac{\mathrm{Y}}{d_x^2 y} \ldots\ldots (1),$$

$$\therefore \frac{2 d_x^2 t}{d_x t^3} = d_x\left(\frac{\mathrm{Y}}{d_x^2 y}\right),$$

$$\therefore \mathrm{R} = \frac{d_x s\, d_x^2 t}{d_x t^3} = \tfrac{1}{2} d_x s\,.\,d_x\left(\frac{\mathrm{Y}}{d_x^2 y}\right) \ldots\ldots (2).$$

Cor. 1. If the resistance ∝ (density) . (velocity)^2 = $\mathrm{Q} d_t s^2$

$$\mathrm{R} = \mathrm{Q} d_t s^2 = \frac{\mathrm{Q} d_x s^2}{d_x t^2} = -\mathrm{QY}\,.\,\frac{d_x s^2}{d_x^2 y}, \text{ from (1),}$$

$$\therefore \mathrm{Q} = -\frac{\mathrm{R}}{\mathrm{Y}} \cdot \frac{d_x^2 y}{d_x s^2} = -\tfrac{1}{2} \cdot \frac{d_x^2 y}{\mathrm{Y} d_x s} \cdot d_x\left(\frac{\mathrm{Y}}{d_x^2 y}\right).$$

Cor. 2. On the same supposition, having the density to find the force.

$$-2\mathrm{Q}d_xs = \frac{d_x^2y}{\mathrm{Y}} \cdot d_x\left(\frac{\mathrm{Y}}{d_x^2y}\right),$$

$$\therefore \mathrm{Y} = e^{-2\int_s\mathrm{Q}} \,.\, d_x^2y.$$

Cor. 3. The velocity is that due to ¼ chord of curvature.

$$\text{For (vel.)}^2 = d_ts^2 = \frac{d_xs^2}{d_xt^2} = -\frac{\mathrm{Y}d_xs^2}{d_x^2y} = \mathrm{Y}d_sx \,.\, \frac{d_xs^3}{-d_x^2y}$$

$= 2\mathrm{Y}$. ¼ chord of curvature.

∴ the space due to the velocity = ¼ chord of curvature.

194. To find the same as in last article, the force being situated in a fixed centre.

Using the notation of Art. 87, we have

$$d_t^2x = -\mathrm{F} \,.\, \frac{x}{r} - \mathrm{R}d_tx,\quad d_t^2y = -\mathrm{F} \,.\, \frac{y}{r} - \mathrm{R}d_ty,$$

$$\therefore d_ty\, d_t^2x - d_tx\, d_t^2y = \frac{\mathrm{F}}{r} \cdot (yd_tx - xd_ty) = \mathrm{F} \,.\, \frac{pd_ts}{r}.$$

$$\therefore \frac{d_ts^3}{\text{rad. curv.}}\left(\text{which} = \frac{d_ts^3}{rd_pr}\right) = \mathrm{F} \,.\, \frac{pd_ts}{r}.$$

$$\therefore d_ts^2 = \mathrm{F}pd_pr = 2\mathrm{F} \,.\, \tfrac{1}{4}\text{ chord of curvature.}$$

Hence the velocity is that due to ¼ chord of curvature.

Again,

$$yd_t^2x - xd_t^2y = -\mathrm{R}\,(yd_tx - xd_ty) = -\mathrm{R}p.$$

$$\therefore -2\mathrm{R}p^2 = 2p\,(yd_t^2x - xd_t^2y) = 2pd_ts \,.\, d_s\,(yd_tx - xd_ty)$$

$$= 2pd_ts \,.\, d_s(pd_ts) = d_s(p^2d_ts^2) = d_s(\mathrm{F}p^3d_pr).$$

$$\therefore \mathrm{R} = -\frac{d_s(\mathrm{F}p^3d_pr)}{2p^2}.$$

Cor. 1. If the resistance ∝ (density) . (velocity)^2 = $\mathrm{Q}d_ts^2$.

$$\therefore \mathrm{R} = \mathrm{Q}d_ts^2 = \mathrm{QF} \,.\, pd_pr,$$

$$\therefore \frac{d_s(\mathrm{F}p^3d_pr)}{2\mathrm{F}p^3d_pr} = -\mathrm{Q}.$$

Cor. 2. On the same supposition, having the density to find the force.

From the last equation $\mathrm{F} = e^{-2\int_s \mathrm{Q}} \,.\, \frac{d_r p}{p^3}$.

Cor. 3. On the same supposition, having the density and force to find the orbit.

Taking the original equations

$$d_t^2 x = -\mathrm{F}\,.\,\frac{x}{r} - \mathrm{R}d_t x,\ d_t^2 y = -\mathrm{F}\,.\,\frac{y}{r} - \mathrm{R}d_t y,\ \mathrm{R}=\mathrm{Q}\,.\,d_t s^2,$$

we obtain,

$$d_t y\ d_t^2 x - d_t x\ d_t^2 y = \frac{\mathrm{F}}{r}(y d_t x - x d_t y) = \mathrm{F}\,.\,\frac{p d_t s}{r},$$

$$\therefore \frac{d_t s^3}{r d_p r} = \mathrm{F}\,.\,\frac{p d_t s}{r},\ \text{and}\ \therefore d_t s^2 = \mathrm{F} p d_p r.$$

Again,

$$y d_t^2 x - x d_t^2 y = \mathrm{R} d_s t\,.\,(y d_t x - x d_t y) = \mathrm{Q} d_t s\,.\,(y d_t x - x d_t y).$$

Hence, by integration,

$$h e^{-\int_s \mathrm{Q}} = y d_t x - x d_t y = p d_t s,$$

$$\therefore \mathrm{F} p^3 d_p r = p^2 d_t s^2 = h^2 e^{-2\int_s \mathrm{Q}},$$

$$\text{and} \therefore \frac{\mathrm{F} e^{2\int_s \mathrm{Q}}}{h^2 u^2} = \frac{-d_u p}{p^3} = \tfrac{1}{2}\, d_\theta \left(\frac{1}{p^2}\right) . d_u \theta,$$

$$= \tfrac{1}{2}\, d_\theta (u^2 + d_\theta u^2) . d_u \theta,$$

$$= u + d_\theta^2 u.$$

This result might have been deduced at once from Cor. 2.

Cor. 4. Hence, besides the expression in Cor. 2 for F, we have also

$$\mathrm{F} = h^2 u^2 (u + d_\theta^2 u) e^{-2\int_s \mathrm{Q}}.$$

195. To determine the same as in Art. 112, the motion taking place in a medium, the resistance of which varies as the velocity.

Using the same notation and Fig. as in that article, and supposing the resistance $= 2\mathrm{M}kgv = 2\mathrm{M}kgd_ts$, we have

$$d_t^2s = -\frac{gs}{a} - 2kgd_ts, \text{ or } d_t^2s + 2kgd_ts + \frac{g}{a}.s = 0,$$

which is true both of the ascent and descent;

$$\therefore s = e^{-kgt}.(\mathrm{C}\sin ht + \mathrm{C}'\cos ht), \text{ where } h^2 = \frac{g}{a} - k^2g^2;$$

and $d_ts = e^{-kgt}.\{(\mathrm{C}h - \mathrm{C}'kg)\cos ht - (\mathrm{C}kg + \mathrm{C}'h)\sin ht\}$.

But when $t = 0$, $s = \beta$, and $d_ts = 0$, $\therefore \mathrm{C}' = \beta$, $\mathrm{C} = \frac{\mathrm{B}kg}{h}$,

$$\text{and } \therefore s = \frac{\beta}{h}.(kg\sin ht + h\cos ht).e^{-kgt}, \quad \ldots\ldots (1),$$

$$\text{and } d_ts = -\frac{\beta g}{ah}.e^{-kgt}.\sin ht \ldots\ldots\ldots\ldots\ldots (2).$$

Now at the beginning and end of each oscillation $d_ts = 0$, and $\therefore \sin ht = 0 = \sin\pi = \sin 2\pi = \sin 3\pi = \ldots;$ and $\therefore h.(\text{time of an oscillation}) = \pi$,

$$\therefore \text{time of oscillation} = \frac{\pi}{\sqrt{\frac{g}{a} - k^2g^2}},$$

which is independent of the magnitude of the arc of oscillation; and $\therefore$ the cycloid is *tautochronous* in this medium as well as in a vacuum.

COR. 1. The velocity will be a maximum when $-kgt + \log_e ht$ is a maximum, or when

$$\frac{h}{kg} = \tan ht, \text{ by } \textit{Diff. Calc.}, \text{ or when}$$

$$s = \frac{\beta kg}{\sqrt{\frac{g}{a} - k^2g^2}}.e^{-\frac{kg}{h}.\tan^{-1}\frac{h}{kg}}.$$

COR. 2. The velocity at the lowest point

$$= \beta\sqrt{\frac{g}{a}}.e^{\frac{kg}{h}.\tan^{-1}\frac{h}{kg}},$$

and the difference between this and the greatest velocity

$$=\beta\sqrt{\frac{g}{a}}\cdot\left(e^{-\frac{kg}{h}.\tan^{-1}\frac{h}{kg}}-e^{\frac{kg}{h}.\tan^{-1}\frac{h}{kg}}\right).$$

COR. 3. By writing π for ht, we have

$$\text{the arc of ascent} = \beta . e^{-\frac{kg\pi}{h}},$$

$\therefore$ the successive arcs of ascent are in geometrical progression, the common ratio being $e^{-\frac{kg\pi}{h}}$.

Hence the whole space described before the motion ceases

$$= \beta + 2\beta e^{-\frac{kg\pi}{h}} + 2\beta e^{-\frac{2kg\pi}{h}} + \dots$$

$$= \beta . \frac{1 + e^{-\frac{kg\pi}{h}}}{1 - e^{-\frac{kg\pi}{h}}}.$$

196. To determine the same as in the last article, the resistance varying as (vel.)^2.

Let resistance $= \mathrm{M}kgv^2$;

$\therefore\ - vd_sv = \frac{gs}{a} - kgv^2$, both for the ascent and descent.

$$\therefore v^2 . e^{-2kgs} = -\frac{2g}{a}.\int_s se^{-2kgs} = \frac{2kgs+1}{2k^2ga}\ e^{-2kgs} + \mathrm{C}.$$

But when $s = \beta$, $v = 0$,

$$\therefore\ v^2 = \frac{2kgs+1}{2k^2ga} - \frac{2kg\beta+1}{2k^2ga}.e^{-2kg(\beta-s)}.$$

197. To determine the motion of a body descending in a circular arc, acted on by gravity in a medium, the resistance of which varies as the square of the velocity.

Using the notation of Art. 113, and putting v for the velocity at P, we have the accelerating force of gravity down the curve $= g . \sin\frac{s}{a}$.

Let the resistance $= \mathrm{M}kgv^2$;

$\therefore v d_s v = -g \cdot \sin \frac{s}{a} + k g v^2$, both for the ascent and descent; which being a linear equation, its integral is

$$(1 + 4a^2k^2g^2) \cdot v^2 = 2ag \cdot \left(\cos \frac{s}{a} + 2akg \cdot \sin \frac{s}{a}\right)$$

$$- 2ag\, e^{-2kg(\beta - s)} \cdot \left(\cos \frac{\beta}{a} + 2akg \cdot \sin \frac{\beta}{a}\right),$$

β being the value of s when the motion commences.

COR. 1. If the value of β be such that $\cos \frac{\beta}{a} + 2akg \cdot \sin \frac{\beta}{a} = 0$;

$$\text{or, } 2akg = \tan \left(\frac{\beta}{a} - \frac{\pi}{2}\right),$$

$$\text{then } v^2 = 2ag \sin \frac{\beta}{a} \cdot \sin \left(\frac{\beta}{a} - \frac{s}{a}\right).$$

Now if the $\angle$ BAC $= \frac{\beta}{a}$ (Fig. 62), and a body begins to descend in a vacuum from C, acted on by an accelerating force $= g \sin \frac{\beta}{a}$ in the direction AF, the (vel.)^2 at P $= 2g \sin \frac{\beta}{a} . a \sin \text{PAC} = 2ag \sin \frac{\beta}{a} \cdot \sin \left(\frac{\beta}{a} - \frac{s}{a}\right)$ = (vel.)^2 of the body in the resisting medium, and hence the motion of the body in the medium will be exactly the same as that of the body in the vacuum.

COR. 2. When the oscillations and density of the medium are very small, to find the time of an oscillation.

We may assume $s = x\beta + y\beta^2 + \&c.$, where $x, y, \ldots$ are certain functions of t; and since the oscillations are very small, neglecting $\beta^3, \beta^4, \ldots$ we have

$$d_t^2 s = \beta d_t^2 x + \beta^2 d_t^2 y, \ \sin \frac{s}{a} = \frac{\beta}{a} . x + \frac{\beta^2}{a} \cdot y, \ d_t s^2 = \beta^2 d_t x^2;$$

and $\therefore$ the original equation of the motion, which may be put under the form

$$d_t^2 s = -g \sin\frac{s}{a} + kg\,d_t s^2,$$

by substitution is reduced to

$$\left(d_t^2 x + \frac{g}{a}.x\right).\beta + \left(d_t^2 y + \frac{g}{a}\cdot y - kg\,d_t x^2\right)\beta^2 = 0;$$

an equation which must hold good whatever be the value of β, providing it be small.

$$\therefore\ d_t^2 x + \frac{g}{a}.x = 0,\ d_t^2 y + \frac{g}{a}.y - kg\,d_t x^2 = 0.$$

The first gives at once

$$x = \cos\left(t\sqrt{\frac{g}{a}}\right), \text{ and } \therefore\ d_t x = -\sqrt{\frac{g}{a}}.\sin\left(t\sqrt{\frac{g}{a}}\right);$$

the constants introduced by integration being determined from the conditions $t = 0$, $s = \beta$, $d_t s = 0$, or $t = 0$, $x = 0$, $d_t x = 0$.

Substituting for $d_t x$ in the second equation, it becomes

$$d_t^2 y + \frac{g}{a}.y = \frac{kg^2}{2a} - \frac{kg^2}{2a}.\cos\left(2t\sqrt{\frac{g}{a}}\right).$$

$$\therefore\ y = -\frac{2kg}{3}.\cos\left(t\sqrt{\frac{g}{a}}\right) + \frac{kg}{2} + \frac{kg}{6}.\cos\left(2t\sqrt{\frac{g}{a}}\right);$$

the constants introduced by integration being determined from the conditions $t = 0$, $y = 0$, $d_t y = 0$.

$$\therefore\ \frac{s}{\beta} = \left(1 - \frac{2kg\beta}{3}\right).\cos\left(t\sqrt{\frac{g}{a}}\right) + \frac{kg\beta}{2} + \frac{kg\beta}{6}\cos\left(2t\sqrt{\frac{g}{a}}\right) \quad . \ . \ (1),$$

$$\therefore\ \frac{d_t s}{\beta} = -\left(1 - \frac{2kg\beta}{3}\right).\sqrt{\frac{g}{a}}.\sin\left(t\sqrt{\frac{g}{a}}\right) - \frac{kg\beta}{3}.\sqrt{\frac{g}{a}}.\sin\left(2t\sqrt{\frac{g}{a}}\right).$$

At the end of an oscillation $d_t s = 0$, and since the second term of the above expression is very small, $t\sqrt{\frac{g}{a}} = \pi$ very nearly; and $\therefore \sin\left(2t\sqrt{\frac{g}{a}}\right)$ is very small, and being multiplied by $kg\beta$ may be neglected, as being of the third order.

$$\therefore\ \sin\left(t\sqrt{\frac{g}{a}}\right) = 0 = \sin\pi,$$

$$\therefore t = \pi \sqrt{\frac{a}{g}},$$

and ∴ the time of an oscillation is unaltered by the resistance. (Art. 151).

COR. 3. On the same supposition, to find the time of descending to the lowest point.

Let t' be the time required; then ∵ it is very nearly $= \frac{\pi}{2} \sqrt{\frac{a}{g}}$, assume

$$t' = \left(\frac{\pi}{2} + x\right) \cdot \sqrt{\frac{a}{g}},$$

and substitute in equation (1) making $s = 0$, and neglecting powers of x higher than the first,

$$\therefore x = \tfrac{1}{3} kg\beta.$$

$$\therefore t' = \frac{\pi}{2} \cdot \sqrt{\frac{a}{g}} + \frac{k\beta\sqrt{ag}}{3}.$$

Hence the time of the ascent

$$= \frac{\pi}{2}\sqrt{\frac{a}{g}} - \frac{k\beta\sqrt{ag}}{3}.$$

COR. 4. On the same supposition, to find the length of an entire oscillation.

In equation (1) writing $\pi \sqrt{\frac{a}{g}}$ (the time of an oscillation) for t, we find the arc of ascent

$$= \beta - \tfrac{4}{3} kg\beta^2;$$

consequently the decrement $= \frac{4}{3} kg\beta^2$.

198. The following problem, which is of great utility in determining the effect of a small disturbing force upon a clock pendulum, is added here on account of its importance, although

only a particular case of it properly falls under the subject of this chapter.

A particle oscillates in a cycloid, as in Art. 67, and is acted on by a small disturbing force; to determine its effect on the time and arc of vibration.

Using the notation of Art. 195, let f be the disturbing force acting in the direction of a tangent to the cycloid.

Then

$$d_t^2 s = -\frac{g}{a} \cdot s + f, \text{ or } d_t^2 s + n^2 s = f \; \ldots\ldots \; (1),$$

by writing $n^2 a$ for g.

Now if there were no disturbing force, or if f were $= 0$, we should, by integration, find

$$s = \beta \sin (nt - \text{C});$$

β being the length of the semi-arc of vibration, and $\frac{\text{C}}{n}$ the time of passing the lowest point.

Assuming this equation, therefore, as the proper form of the integral of equations (1) we shall have, by the variation of the parameters β, C, the following equations:

$$s = \beta \sin (nt - \text{C}), \; \ldots\ldots\ldots\ldots \; (2),$$

$$d_t s = n\beta \cos (nt - \text{C}), \; \ldots\ldots\ldots\ldots \; (3),$$

$$0 = \sin (nt - \text{C}) \,.\, d_t\beta - \beta \cos (nt - \text{C}) \,.\, d_t\text{C},$$

$$f = n \cos (nt - \text{C}) \,.\, d_t\beta + n\beta \sin (nt - \text{C}) \,.\, d_t\text{C}.$$

The two last give

$$d_t\beta = \frac{f}{n} \,.\, \cos (nt - \text{C}), \;\; d_t\text{C} = \frac{f}{n\beta} \,.\, \sin (nt - \text{C}), \ldots (4);$$

and if we could integrate these two equations, the motion would be completely determined; but their forms are such that they do not often admit of integration.

Since, however, f is but small, an approximation sufficiently exact may be obtained by integrating the expressions (4) on the supposition that β and C are constant; for as their variation depends on f, the errors introduced by supposing them constant will depend on f^2, or on quantities of that order, and may be neglected, since f is small.

From (1) and (2) we see that the time of arriving at the highest point will be determined by assuming $\cos(nt - C) = 0$, and at the lowest point $\sin(nt - C) = 0$; and the alteration in the length of the arc of vibration, which takes place in one oscillation, will be determined from (4), by integrating $\frac{f}{n} \cdot \cos(nt - C)$, through the limits of t, corresponding to one oscillation; or from that value of t which gives $nt - C = \frac{\pi}{2}$, to that which gives $nt - C = \frac{3\pi}{2}$; hence the decrement of the arc of vibration for one oscillation

$$= \frac{1}{n} \cdot \int_t f \cos(nt - C),$$

taken between

$$t = \frac{3\pi - 2C}{2n} = t'', \text{ and } t = \frac{\pi - 2C}{2n} = t'.$$

Let C'', C' be the values of C at these limits of t; then

$$nt'' - C'' = \frac{3\pi}{2}, \text{ and } nt' - C' = \frac{\pi}{2},$$

$$\therefore t'' - t' = \frac{\pi + C'' - C'}{n}.$$

Now $C'' - C' = \int_t d_t C = \frac{1}{n\beta} \cdot \int_t f \sin(nt - C)$, between the same limits; and $\therefore$ the time of a vibration $= t'' - t'$ $= \frac{\pi}{n} + \frac{1}{n^2\beta} \cdot \int_t f \sin(nt - C)$.

Hence

the alteration in the arc of vibration $= \frac{1}{n} \cdot \int_t f \cos(nt - C)$,

and . . . . time . . . . $= \frac{1}{n^2\beta} \cdot \int_t f \sin(nt - C)$;

both integrals being taken between the limits above stated.

The same formulæ may be otherwise expressed, if f be given in terms of s; for

$$\frac{1}{n} \cdot \int_t f \cos(nt - \mathrm{C}) = \frac{1}{n} \cdot \int_t \frac{f d_t s}{n\beta} = \frac{1}{n^2\beta} \cdot \int_t f.$$

and $\frac{1}{n^2\beta} \cdot \int_t f \sin(nt - \mathrm{C}) = \frac{1}{n^2\beta} \cdot \int_t f \sin(nt - \mathrm{C})\, d_t t$

$$= \frac{1}{n^3\beta^3} \cdot \int_t f \tan(nt - \mathrm{C}) = \frac{1}{n^3\beta^3} \cdot \int_t \frac{sf}{\sqrt{\beta^2 - s^2}}.$$

199. As an application of these principles, suppose the resistance varies as the m^{th} power of the velocity, or $= \mathrm{M}kgv^m$; m being any integer.

Here $f = -kgn^m\beta^m . \cos^m(nt - \mathrm{C})$, from (3).

Hence the alteration in the arc of semi-vibration $= -kgn^{m-1}\beta^m . \int_t \cos^m(nt - \mathrm{C})$,

$\left(\text{taken between the limits } nt - \mathrm{C} = -\frac{\pi}{2} \text{ to } nt - \mathrm{C} = \frac{\pi}{2}\right)$,

$$= \frac{m.(m-2) \; . \; . \; . \; . \; 1}{(m+1)(m-1) \; . \; . \; 2} . \, kgn^{m-2}\beta^m\pi, \text{ when } m \text{ is odd; . . (5),}$$

and

$$= \frac{m.(m-2) \; . \; . \; . \; . \; 2}{(m+1).(m-1) \; . \; . \; 3} . \, 2kgn^{m-2}\beta^m, \text{ when } m \text{ is even, . . (6).}$$

$$\text{When } m = 2, \text{ this } = \frac{4}{3} . \, kg\beta^2.$$

The alteration in the time of vibration $= kgn^{m-2}\beta^{m-1} . \int_t \cos^m(nt - \mathrm{C}) \sin(nt - \mathrm{C})$, which between the same limits $= 0$, whether m be whole or fractional.

The reader will find many other applications of these formulæ in a Paper (by Professor Airy) of the Cambridge Phil. Trans. vol. 3, from which the whole of Arts. 197, 198, has been extracted with slight modifications. See also Airy's *Planetary Theory*, prop. 38, 39.

200. Cor. When the resistance is any function of the velocity, the alteration in the time of oscillation $= 0$. For, in this case, the general term in the expression for f would be multiplied by $\cos^m(nt - \mathrm{C})$, which produces no alteration in C.

201. As a further application and extension of these formulæ, suppose that a particle acted on by a force tending to a fixed centre, and varying as $\frac{1}{D^2}$, moves in a medium of small density, the resistance of which varies as the (density). (vel.)^2; to find the alteration in the eccentricity, and in the position of the axis major.

The general equation of Cor. 3, Art. 194, becomes, in this case,

$$d_\theta^2 u + u = \frac{\mu}{h^2} . e^{2\int_s Q},$$

which may be written

$$d_\theta^2 \left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = \frac{\mu}{h^2} \left(e^{2\int_s Q} - 1\right).$$

Comparing this with equation (1), we find

$$u - \frac{\mu}{h^2} = s, \; n = 1, \; t = \theta, \; f = \frac{\mu}{h^2} \cdot \left(e^{2\int_s Q} - 1\right),$$

and equations (2) and (3) become

$$u = \frac{\mu}{h^2} + \beta \sin(\theta - C),$$

$$d_\theta u = \beta \cos(\theta - C);$$

and at perigee $\theta - C = \frac{\pi}{2}$, and at apogee $\theta - C = \frac{3\pi}{2}$; we see therefore that the limits of θ are the same as those of t in the general formulæ;

hence the variation in $\frac{h^2 \beta}{\mu}$, that is, in the eccentricity,

$$= \frac{h^2}{\mu} . \int_\theta f \cos(\theta - C);$$

and in C, that is, in the position of the major axis,

$$= \frac{1}{\beta} . \int_\theta f \sin(\theta - C).$$

Now $f = \frac{\mu}{h^2} (e^{2\int_s Q} - 1) = \frac{\mu}{h^2} \cdot 2\int_s Q$, nearly $= \frac{2\mu}{h^2} . Qs$ nearly;

and s being a function of $\theta - C$, we may assume

$$f = \frac{2\mu Q}{h^2} \cdot \{a_0 + a_1 \cos(\theta - C) + a_2 \cos^2(\theta - C) + . . \}:$$

and $\because$ Q is nearly constant, each of these produces no effect (last Cor.) on C, and $\therefore$ the axis major is invariable in position.

The general term for the variation of the eccentricity is

$$2Qa_{m-1}\int_\theta \cos^m(\theta - C);$$

the value of which between the proper limits is known from (5) and (6).

Since β in the last example is always diminished, it will be so in this, the two being exactly parallel; and $\therefore$ the eccentricity is always *diminished* by a resisting medium.

Other Problems might be added, but those given above will be found sufficient to elucidate the Principles and Formulæ peculiar to this part of our subject.

APPENDIX.

We may also arrive at the equations of motion in Art. 80, by the following method, which is given lest the demonstration in the text should be thought too brief, and not sufficiently evident to minds not much accustomed to Dynamical reasoning.

Let X, Y, Z be the resolved parts, parallel to rectangular co-ordinate axes, of all the forces acting upon the particle m at the point P, (the co-ordinates of which are $x\,y\,z$) its place at the time t reckoning from some fixed epoch; $\alpha\,\beta\,\gamma$ the $\angle^s$ which the direction of its motion when at P makes with the co-ordinate axes; v its velocity; F the resultant of all the forces acting upon it, and $\phi\,\chi\,\psi$ the $\angle^s$ which F's direction makes with the axes. Then, since the place of the body depends on the time it has been in motion, the co-ordinates of Q, the place of m at the time $t + h$, will be

$$x + d_t x \,.\, h + d_t^2 x \,.\, \frac{h^2}{1\,.\,2} + \ldots$$

$$y + d_t y \,.\, h + d_t^2 y \,.\, \frac{h^2}{1\,.\,2} + \ldots$$

$$\text{and} \quad z + d_t z \,.\, h + d_t^2 z \,.\, \frac{h^2}{1\,.\,2} + \ldots$$

Now when m is at P its motion is influenced by its own inertia and the force F, and by the Law of the Composition of Forces (Art. 17) we may suppose the effect of each to take place separately for the very small time h. The former force (inertia) will cause the particle to move over the spaces

$$v \cos\alpha \,.\, h, \quad v \cos\beta \,.\, h, \quad v \cos\gamma \,.\, h,$$

in directions of the co-ordinates; and, if the latter force (F) were continued uniform, it would cause it to move in the direction in which it acts through the space $\frac{1}{2}\mathrm{F}.h^2$, and therefore, through the spaces

$$\tfrac{1}{2}\mathrm{F}h^2.\cos\phi\left(=\mathrm{X}.\frac{h^2}{2}\right),\quad \tfrac{1}{2}\mathrm{F}h^2.\cos\chi\left(=\mathrm{Y}.\frac{h^2}{2}\right),$$

$$\tfrac{1}{2}\mathrm{F}h^2.\cos\psi\left(=\mathrm{Z}.\frac{h^2}{2}\right),$$

in the directions of the co-ordinates; and ∴ the co-ordinates of the point Q′, where the particle would be situated at the time $t+h$, if F were continued uniform, are

$$x+v\cos\alpha.h+\mathrm{X}.\frac{h^2}{1.2},$$

$$y+v\cos\beta.h+\mathrm{Y}.\frac{h^2}{1.2},$$

$$z+v\cos\gamma.h+\mathrm{Z}.\frac{h^2}{1.2}.$$

But since the force F, as we diminish h, approaches towards uniformity, and *is* ultimately uniform, ∴ ultimately Q and Q′ coincide; and ∴ equating the coefficients of h in the above expressions for their co-ordinates, we obtain

$d_t x = v\cos\alpha =$ velocity in direction of x,

$d_t y = v\cos\beta =$ . . . . of y,

$d_t z = v\cos\gamma =$ . . . . of z,

and

$d_t^2 x = \mathrm{X}$ = resolved force in direction of x,

$d_t^2 y = \mathrm{Y}$ = ... of y,

$d_t^2 z = \mathrm{Z}$ = ... of z.

A SHORT TREATISE

ON

ATTRACTIONS.

ATTRACTION.

1. **Kepler** first discovered, with much labour of observation and calculation, that the Planets describe about the Sun areas proportional to the times of their description; that their orbits are ellipses; and that the squares of their periods are proportional to the cubes of the major axes. The first of these laws, compared with *Dynamics*, Art. 89, shews that the force which acts on each planet is directed towards the centre of the Sun; the second, compared with Art. 90, Ex. 3, that for the same planet this force varies inversely as the square of the distance between the centres of the Sun and planet; from the third, compared with Art. 96, Cor., we conclude that it is the same force which acts upon all the planets, so that if they were all removed to the same distance from the Sun's centre, they would fall towards it with the same velocity; and, therefore, the respective gravitating forces of the planets are proportional to their masses divided by the squares of their distances from his centre. The Laws of Kepler are equally applicable to the motion of satellites about their primaries; and therefore the Sun, and such of the planets as have satellites, exert attractive influences which are directed towards their centres, and vary in intensity in the inverse ratio of the squares of the distances from their centres. Analogy leads us to conclude that the other planets exert similar influences, varying according to the same law. Sir Isaac Newton proved that the Moon is retained in her orbit by the same force as that which causes the descent of bodies near the Earth's surface. It is

therefore probable, that every particle of matter is endued with the power of attracting every other particle, and that the attraction of the Sun and planets is the resultant of the attractions of their several particles. It will be shewn hereafter, that if this be true, each particle must exert an attractive influence, the intensity of which varies according to the same law as that exerted by the Sun and planets. Every molecule, therefore, in the solar system, attracts every other with a force varying as *the number of its particles* (that is, *its mass*) *directly*, and the *square of their distance inversely:* and this is the law of attraction, which, with a single exception in the case of a sphere, will always be supposed to be observed by small elements of matter. On the subject of this article the reader may consult the third book of Newton's *Principia;* chap. I. liv. ii. of the *Mécanique Céleste*; and the *Exposition du Système du Monde* of Laplace; where he will find the arguments stated at considerable length.

2. To find the attraction of a thin prism, of uniform density, on a given particle.

Let A (Fig. 1) be the attracted particle, BC the given straight rod. Draw AD $\perp$ BC, and let AD $= f$, DP $= y$, PQ $= \delta y$, $\rho =$ the density of the rod, S $=$ the area of a section of it, which is supposed to be indefinitely small; F, G the resolved parts of the attraction of PD in the directions AD, DB.

Then the mass of PQ $= \rho S \delta y$, and $AP^2 = f^2 + y^2$, $\therefore$ the whole attraction of PQ in the direction AP is ultimately equal to

$$\frac{\rho S \delta y}{f^2 + y^2},$$

and $\therefore$ $\delta F =$ that part of it which acts in the direction AD

$$= \frac{\rho S \delta y}{f^2 + y^2} \cdot \frac{AD}{AP} = \frac{\rho S f \delta y}{(f^2 + y^2)^{\frac{3}{2}}},$$

$$\therefore\ d_y\mathrm{F} = \frac{\rho \mathrm{S} f}{(f^2+y^2)^{\frac{3}{2}}},$$

Similarly, $\delta\mathrm{G} = \frac{\rho\mathrm{S}\delta y}{f^2+y^2}\cdot\frac{\mathrm{PD}}{\mathrm{AP}} = \frac{\rho\mathrm{S}y\delta y}{(f^2+y^2)^{\frac{3}{2}}}$,

$$\therefore\ d_y\mathrm{G} = \frac{\rho\mathrm{S}y}{(f^2+y^2)^{\frac{3}{2}}}.$$

Hence, by integration,

$$\mathrm{F} = \frac{\rho\mathrm{S}y}{f\sqrt{f^2+y^2}} + \mathrm{C},\quad \mathrm{G} = \frac{-\rho\mathrm{S}}{\sqrt{f^2+y^2}} + \mathrm{C}',$$

and therefore, for the whole rod,

$$\mathrm{F} = \frac{\rho\mathrm{S}}{f}\cdot(\sin\mathrm{DAC} + \sin\mathrm{DAB});\quad \mathrm{G} = \rho\mathrm{S}\,.\left(\frac{1}{\mathrm{AB}} - \frac{1}{\mathrm{AC}}\right).$$

3. Cor. With the centre A, and radius AD, describe an arc of a circle, cutting the lines AB, AC, AP, AQ, in b, c, p, q; and suppose $b\mathrm{D}c$ a material arc of the same density and thickness as the rod AB.

Then the attraction of PQ in the direction AP $= \frac{\rho\mathrm{S}\,.\,\mathrm{PQ}}{\mathrm{AP}^2}$.

But drawing $\mathrm{P}n \perp \mathrm{AP}$, we have by the similar $\triangle^s$ APD, $\mathrm{PQ}n$,

$$\frac{\mathrm{PQ}}{\mathrm{AP}} = \frac{\mathrm{P}n}{\mathrm{AD}};$$

and by the similar $\triangle^s$ $\mathrm{A}pq$, $\mathrm{AP}n$, we have

$$\frac{\mathrm{P}n}{\mathrm{AP}} = \frac{pq}{\mathrm{AD}},\ \text{and}\ \therefore\ \frac{\mathrm{PQ}}{\mathrm{AP}^2} = \frac{\mathrm{P}n}{\mathrm{AD}\,.\,\mathrm{AP}} = \frac{pq}{\mathrm{AD}^2};$$

$\therefore$ the attraction of PQ in direction AP $= \frac{\rho\mathrm{S}\,.\,pq}{\mathrm{AD}^2}$ = the attraction of the circular arc pq.

Hence the particle A is attracted by the straight line BC in the same manner as by the circular arc bc; and therefore A is attracted in a direction bisecting the $\angle$ BAC.

Prob. Two straight rods, situated in the same straight line, are mutually attracted towards each other; to determine the force necessary to keep them asunder.

By Art. 2, the attraction of the rod AB (Fig. 2) whose density $= \rho$, and of which a section $=$ S, is $\rho \mathrm{S}\left(\frac{1}{\mathrm{BP}} - \frac{1}{\mathrm{AP}}\right)$ on a unit at P; and $\therefore$ the weight of the element PQ

$$= \rho \mathrm{S}\left(\frac{1}{\mathrm{BP}} - \frac{1}{\mathrm{AP}}\right).\rho'\mathrm{S}'\delta x,$$

$$= \rho\rho'\mathrm{SS}'\left(\frac{\delta x}{x} - \frac{\delta x}{\mathrm{AB}+x}\right),$$

where ρ', S′ are the density and area of a section of the rod CD, and $x = \mathrm{BP}$.

$$\therefore \text{ weight of CP} = \rho\rho'\mathrm{SS}'\int_x\left(\frac{1}{x} - \frac{1}{\mathrm{AB}+x}\right),$$

$$= \rho\rho'\mathrm{SS}'.\log_e\frac{x}{\mathrm{AB}+x} + \mathrm{C}.$$

$$\therefore \text{ weight of CD} = \rho\rho'\mathrm{SS}'.\log_e\left(\frac{\mathrm{BD}\,.\,\mathrm{AC}}{\mathrm{BC}\,.\,\mathrm{AD}}\right).$$

Similarly, weight of AB $= \rho\rho'\mathrm{SS}'.\log_e\left(\frac{\mathrm{BD}\,.\,\mathrm{AC}}{\mathrm{BC}\,.\,\mathrm{AD}}\right)$, by changing B into D, and C into A, and conversely: which is equal to the weight of CD, as it ought to be, from Art. 22, *Dynamics*.

4. To find the attraction of a circular plate, of uniform density and of inconsiderable thickness, on a particle situated in a line through its centre perpendicular to its plane.

Let t be the thickness of the plate, f the distance of the attracted particle from its centre. Then the mass of a concentric circular annulus, whose radii are r, $r+\delta r$, is ultimately equal to $2\pi\rho t r\delta r$, and the attraction (δF) of this on the given particle, which manifestly takes place in direction of f,

$$= \frac{2\pi\rho t f r\delta r}{(f^2 + r^2)^{\frac{3}{2}}};$$

$$\therefore \text{F} = \int_r \frac{2\pi\rho t f r}{(f^2 + r^2)^{\frac{3}{2}}} = 2\pi\rho t . \left(1 - \frac{f}{\sqrt{f^2 + r^2}}\right).$$

5. Cor. 1. The attraction of all circular plates, of the same thickness and density, on a particle situated as in the proposition, will be the same, if their apparent magnitudes, as seen from the attracted point, be the same.

6. Cor. 2. When r is infinite, we have the attraction of a plate of unlimited extent $= 2\pi\rho t$, which is independent of the distance of the particle from it.

7. To determine the attraction of a cone, of uniform density, on a particle at its vertex.

Let 2θ be the vertical $\angle$ of the cone. Then (Art. 4) the attraction of a slice, whose thickness is δf, parallel to the base, at the distance f from the vertex, is ultimately equal to $2\pi\rho\delta f . (1 - \cos\theta)$; and therefore the attraction of the cone $= 2\pi\rho\ (1 - \cos\theta)$. (altitude).

8. To determine the attraction of a spherical shell on a given particle; the attraction of the particles, of which the shell is composed, varying as any function of the distance.

Let S (Fig. 3) be the centre of the shell, p the attracted particle, AB the section of the external surface by the plane of the paper; in this plane draw SP, SP′ making $\angle^s\ \theta,\ \theta + \delta\theta$ with pS; and let SP, SP′ be produced to Q, Q′; a, a' the external and internal radii of the shell, $b = \text{S}p$, $u = p\text{P}$, $r = \text{SP} = \text{SP}'$, $\delta r = \text{PQ} = \text{P}'\text{Q}'$, $\mu\phi u =$ the attraction of a mass μ at distance u, $\rho =$ the density of the shell.

Then $u^2 = b^2 - 2br\cos\theta + r^2 \ldots.$ (1), and PQ′ ultimately $= r\delta\theta\delta r$, and the mass of the annulus generated by the revolution of this element round $\text{S}p = 2\pi\rho r^2\delta r . \sin\theta . \delta\theta$

$= -\ 2\pi\rho r^2\delta r\,.\,\delta\,(\cos\theta) = \frac{2\pi\rho r\delta r\,.\,u\delta u}{b}$, by eliminating $\cos\theta$ by means of (1). And the attraction of this elementary annulus in the direction $pS = \frac{2\pi\rho r\delta r\,.\,\phi u\,.\,u\delta u}{b}\,.\cos SpP$,

$$= \frac{2\pi\rho}{b}\,.\,r\delta r\,.\,u\phi u\,.\,\delta u\left(\frac{b^2 - r^2 + u^2}{2bu}\right).$$

Hence the attraction of the shell, which manifestly takes place in the direction pS,

$$= \frac{\pi\rho}{b^2}\,.\int_r\int_u r\,(b^2 - r^2 + u^2)\,\phi u.$$

The integral, with respect to u, is to be taken from $u = b - r$ to $b + r$, when the particle p is *without* the shell; but from $u = r - b$ to $r + b$ if it be *within* it. The integral, with respect to r, is to be taken from $r = a'$ to a.

9. Ex. 1. When the attractive force of each particle varies as the distance.

In this case $\phi u = u$, and

$$\therefore \int_r\int_u r\,(b^2 - r^2 + u^2)\,\phi u = \int_r r\left\{\tfrac{1}{2}(b^2 - r^2)\,u^2 + \tfrac{1}{4}u^4 + C\right\}\ldots(1),$$

which, whether the particle be *within* or *without* the shell,

$$= \int_r 4b^3r^2 = \tfrac{4}{3}\,b^3r^3 + C = \tfrac{4}{3}\,b^3\,(a^3 - a'^3).$$

and $\therefore$ the attraction of the whole shell

$$= \tfrac{4}{3}\,b\rho\,.\,\pi\,(a^3 - a'^3),$$

$$= b\,.\,(\text{mass of the shell}).$$

10. This result may be arrived at in a more simple manner as follows:

Let MN, mn (Fig. 4) be two sections of the shell, by planes $\perp$ to pS, at equal distances from the centre S; and suppose P an element of the shell. Then the force of P on p in the direction pP = (mass of P) . Pp, which resolved in the direction pS = (mass of P) . pN, and is the same as if it were placed at N; and therefore the whole section attracts with the same force as if it were collected at N; similarly the whole section through m attracts with the same force as if it were collected at n, and $\therefore$ the sum of these attractions

$$= (\text{mass of the section through N or } n).(pn \pm p\text{N}),^{*}$$
$$= (\text{the mass of a section}).\,2p\text{S},$$
$$= (\text{the sum of the sections}).\,p\text{S};$$

and the same being true of each pair of equidistant sections, it is clear the whole attraction of the shell

$$= (\text{mass of the shell}).\,p\text{S}, \text{ as before.}$$

11. Ex. 2. When the attractive force of each particle varies inversely as the square of the distance.

In this case $\phi u = \dfrac{1}{u^2}$, and

$$\therefore \int_r \int_u \frac{r(b^2 - r^2 + u^2)}{u^2} = \int_r \left\{ -\frac{b^2 r - r^3}{u} + ru + \text{C} \right\}$$
$$= 0 \text{ or} \int_r 4r^2,$$
$$= 0 \text{ or } \tfrac{4}{3}(r^3 + \text{C}'),$$
$$= 0 \text{ or } \tfrac{4}{3}(a^3 - a'^3),$$

according as the particle is *within* or *without* the shell.

Hence the attraction in the former case $= 0$, and in the latter

$$= \frac{4\pi\rho}{3b^2}.(a^3 - a'^3) = \frac{\text{mass}}{b^2}.$$

12. The results of the last example, which is the case of Nature, may be arrived at in a different manner.

Using the notation of Art. 8, let $\zeta = \angle\, \text{S}p\text{P}$ (Fig. 3), $\text{B} =$ the attraction of the shell in the direction $p\text{S}$, $m =$ the mass of the annulus generated by $\text{PQ}' = 2\pi\rho r^2 \sin\theta \,.\, \delta r \delta\theta$; $\text{V} = \Sigma\left(\dfrac{m}{u}\right) =$ the sum of the quotients of each particle divided by its distance from p.

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[*] The top or bottom sign is used according as the attracted particle is *without* or *within* the sphere.

Then

$$u^2 = b^2 - 2br\cos\theta + r^2, \quad \text{and} \quad \therefore\ u d_\theta u = br\sin\theta,$$

$$\cos\zeta = \frac{b - r\cos\theta}{u} = d_b u\ ; \quad \text{and } \mathrm{B} = \Sigma\left(\frac{m}{u^2}\cdot\cos\zeta\right),$$

also

$$-d_b\mathrm{V} = -\Sigma\, d_b\left(\frac{m}{u}\right) = \Sigma\left(\frac{m}{u^2}\,.\,d_b u\right) = \Sigma\left(\frac{m}{u^2}\cdot\cos\zeta\right) = \mathrm{B}.$$

Now

$$\mathrm{V} = \Sigma\left(\frac{m}{u}\right) = 2\pi\rho\,.\,\Sigma\left(\frac{r^2\sin\theta\,.\,\delta r\,\delta\theta}{u}\right),$$

$$= 2\pi\rho\,.\int_r\int_\theta\left(\frac{r^2\sin\theta}{u}\right),$$

$$= \frac{2\pi\rho}{b}\cdot\int_r\int_\theta(r d_\theta u), \text{ by eliminating } \sin\theta,$$

$$= \frac{2\pi\rho}{b}\cdot\int_r(ru + \mathrm{C}),$$

$$= \frac{2\pi\rho}{b}\cdot\int_r 2r^2, \text{when the particle is } \textit{without} \text{ the shell,}$$

$$\text{but} = \frac{2\pi\rho}{b}\cdot\int_r 2br, \text{ when it is } \textit{within} \text{ the shell,}$$

$$= \frac{4\pi\rho}{3b}\cdot(a^3 - a'^3), \text{ and } 2\pi\rho\,.\,(a^2 - a'^2) \text{ in the two cases.}$$

Hence

$$\mathrm{B} = -d_b\mathrm{V} = \frac{4\pi\rho}{3b^2}\cdot(a^3 - a'^3) = \frac{\text{mass}}{b^2} \text{ in the former case;}$$

and $= 0$ in the latter case.

13. Cor. 1. Since the attraction of a spherical shell on a particle is the same as if its mass were collected at its centre, both when the force of each particle varies as the distance, and when it varies inversely as the square of the distance, it follows that the attraction of a sphere composed of such particles will be the same as if it were collected at its centre, whether it be homogeneous or heterogeneous, providing the density be the same at all equal distances from the centre. For it may be

considered as made up of an infinite number of concentric spherical shells, of indefinitely small thickness and uniform density, each of which produces the same effect as if it were collected at the centre. See Art. 79, *Dynamics.*

14. Cor 2. When the force of each particle varies inversely as the square of the distance, the attractive force of a homogeneous sphere on a particle on its surface, found from Art. 11, by making $a = b$, and $a' = 0$, is $\frac{4}{3}\pi\rho b$, which therefore varies as the radius.

15. To find the attractive force of a sphere on a particle situated within it.

Conceive a concentric spherical surface described within the sphere, and passing through the attracted particle. The sphere is thus divided into a spherical shell, which, by Art. 11, produces no effect on the particle; and a smaller sphere on the surface of which the particle is situated, and whose attraction, by the last Art. $= \frac{4}{3}\pi\rho\,.$ (radius)

$= \frac{4}{3}\pi\rho\,.$ (distance of particle from the centre).

This is, therefore, the attraction of the whole sphere; and it varies as the distance of the particle from the centre. See Art. 79, *Dynamics*.

16. Cor. Two spheres will attract each other in the same manner as if their masses were collected at their respective centres.

For, by Art. 13, the sphere A will attract each particle of B in the same manner as if its whole mass were collected at its centre: and since, by Art. 23, *Dynamics,* the particle exerts an equal re-action on A, therefore each particle of B acts on A in the same manner as if A were collected at its centre; but B acts on a body, collected at a point, as if it were itself collected at its centre (Art. 13); and, therefore, A and B attract each other in the same manner as if they were respectively collected at their centres.

17. This Cor. is true both when the force of each particle varies as the distance and inversely as the square of the distance. Art. 13.

18. From Art. 11, the attraction of a shell of indefinitely small thickness δa,

$$= \frac{4\pi\rho}{3b^2}\{a^3 - (a - \delta a)^3\} = \frac{4\pi\rho}{b^2} \cdot a^2\delta a, \text{ ultimately.}$$

19. If a cone be described enveloping the shell just mentioned, and having its vertex in the attracted point, it will by the circle of contact divide the shell into two parts, (one convex, and the other concave, towards the attracted particle) whose attractions are equal.

For, by Art. 8, the attraction of such a shell

$$= \frac{\pi\rho}{b^2} \cdot \int_u \frac{a\,\delta a\,(b^2 - a^2 + u^2)}{u^2},$$

$$= \frac{\pi\rho a\,\delta a}{b^2} \cdot \left(\mathrm{C} - \frac{b^2 - a^2}{u} + u\right);$$

and this taken from $u = b - a$ to $u = \sqrt{b^2 - a^2}$, gives

$$\frac{\pi\rho a\delta a}{b^2} \cdot 2a, \text{ or } \frac{2\pi\rho}{b^2} \cdot a^2\delta a$$

for the attraction of the convex part, which is $\frac{1}{2}$ the attraction of the whole shell; and therefore the attractions of the convex and concave parts are equal.

20. Or we may demonstrate the same property thus;

$$\mathrm{V} = \Sigma\left(\frac{m}{u}\right) = 2\pi\rho \,.\, \Sigma\left(\frac{a^2 \sin\theta \,.\, \delta a\,\delta\theta}{u}\right),$$

$$= \frac{2\pi\rho a\delta a}{b} \cdot \int_\theta d_\theta u, \text{ as in Art. 12,}$$

$$= \frac{2\pi\rho a\delta a}{b} . (u + \mathrm{C}).$$

This integral taken between $u = b - a$, and $u = b \sin \alpha$, (2α being the ∠ subtended at the centre of the shell by the convex part of the generating circle) will give

$$\mathrm{V} = \frac{2\pi\rho a\delta a}{b} . (b \sin \alpha - b + a) = 2\pi\rho a\delta a . \left(\sin \alpha - 1 + \frac{a}{b}\right)$$

for the convex part; and, therefore, the attraction of this part

$$= - d_b \mathrm{V} = \frac{2\pi\rho a^2 \delta a}{b^2}, \text{ as before.}$$

21. The following is Laplace's method of finding the attraction of a spherical shell for any law of attraction of its component particles.

Let ϕu be the attraction of a unit of matter at the distance u; then, proceeding as in Art. 12, the attraction of the shell, whose internal radius is r and thickness δr will be found

$$= 2\pi\rho r^2 \delta r . \int_\theta \left(\frac{b - r\cos\theta}{u} . \phi u . \sin\theta\right),$$

$$= 2\pi\rho r^2 \delta r . \int_\theta (\phi u . d_b u . \sin\theta).$$

Let now $\phi u . d_b u = d_b \phi_1 u$, then the attraction

$$= 2\pi\rho r^2 \delta r . \int_\theta (d_b \phi_1 u . \sin\theta),$$

$$= 2\pi\rho r^2 \delta r \ . d_b \int_\theta (\phi_1 u . \sin\theta),$$

$$= 2\pi\rho r \delta r . d_b \int_\theta \frac{u\phi_1 u . d_\theta u}{b}, \text{ by eliminating } \sin\theta;$$

$$= 2\pi\rho r \delta r . d_b \int_u \frac{u\phi_1 u}{b};$$

and if the integral of $u\phi_1 u$ be represented generally by $\psi u + \mathrm{C}$, its value between the limits $u = b - r, b + r$, will be

$$\int_u u\phi_1 u = \psi(b + r) - \psi(b - r);$$

and therefore the attraction of the shell on an external particle

$$= 2\pi\rho r \delta r . d_b \left\{\frac{\psi(b + r) - \psi(b - r)}{b}\right\}.$$

Similarly, its attraction on an internal particle

$$= 2\pi\rho r\delta r \, . \, d_b \left\{\frac{\psi(r+b) - \psi(r-b)}{b}\right\}.$$

22. To find according to what law the particles of a spherical shell must attract, in order that the whole shell may attract an external particle in the same manner as if it were collected at its centre.

By the last article, the attraction of the shell

$$= 2\pi\rho r \delta r \, . \, d_b \left\{\frac{\psi(b+r) - \psi(b-r)}{b}\right\}.$$

Now (ψ', ψ'', ... denoting $d_b\psi, d_b{}^2\psi$ . . . .) by expanding these functions, by Taylor's Theorem, the whole expression becomes, (putting M for $4\pi\rho r^2\delta r$ the mass of the shell)

$$= \text{M} \cdot \left\{d_b\left(\frac{\psi' b}{b}\right) + d_b\left(\frac{\psi''' b}{b}\right) \cdot \frac{r^2}{2 \, . \, 3} + \ldots\right\},$$

which will be equal to the attraction of the whole collected at its centre, if

$$d_b\left(\frac{\psi''' b}{b}\right) = 0,$$

for then the terms containing r vanish. Hence, by integration, &c.

$$\psi''' b = 3\text{A}b,$$

$$\therefore \; \psi'' b = \tfrac{3}{2}\,\text{A}b^2 + \text{B},$$

$$\text{and } \psi' b = \tfrac{1}{2}\,\text{A}b^3 + \text{B}b - \text{C}.$$

$$\text{But } \psi' b = b\phi_1 b, \;\; \therefore \; b\phi_1 b = \tfrac{1}{2}\,\text{A}b^3 + \text{B}b - \text{C},$$

$$\therefore \; \phi_1 b = \tfrac{1}{2}\text{A}b^2 + \text{B} - \frac{\text{C}}{b},$$

$$\text{and } \therefore \; \phi b = d_b\phi_1 b = \text{A}b + \frac{\text{C}}{b^2} \cdot$$

that is, the force must vary either as the distance directly, or as the inverse square of the distance, or partly as one and partly as the other.

Hence the law of Nature is the only law of attraction of the particles of matter, that a spherical shell may attract with a force varying inversely as the square of the distance from its centre.

23. To find the law of attraction of each particle, that a spherical shell may exert no attractive force on a particle within it.

In this case, from Art. 21,

$$d_b . \left\{\frac{\psi(r+b)-\psi(r-b)}{b}\right\} = 0,$$

whatever be the value of b; and therefore, by integrating, we must always have

$$\psi(r+b) - \psi(r-b) = -2Cb,$$

or expanding, by Taylor's Theorem,

$$-Cb = \psi' r . b + \psi''' r . \frac{b^3}{1.2.3} + \dots$$

$$\therefore\ \psi' r = -C, \text{ and } \therefore\ -C = \psi' b = b\phi_1 b,$$

$$\therefore\ \phi_1 b = -\frac{C}{b},$$

$$\therefore\ \phi b = d_b \phi_1 b = \frac{C}{b^2},$$

which is the law of Nature; and, therefore, by the last article, the force of each particle varies inversely as the square of the distance.

24. To find the attraction of an oblate spheroid of uniform density and small ellipticity on a particle at its pole.

Let a, b, be the semi major and minor axes of the spheroid, within which inscribe a sphere touching it at its poles. Then the equation of the generating ellipse is

$$y^2 = \frac{a^2}{b^2}(2bx - x^2),$$

the particle being the origin, and the axis of revolution the axis of x.

Then the mass of the circular annulus between the sphere and spheroid, which is contained between planes parallel to the plane of the equator, at distances x and $x + \delta x$ from the particle,

$$= \pi \left(1 - \frac{b^2}{a^2}\right) \rho \, . \, y^2 \, \delta x, \text{ ultimately;}$$

or if $\epsilon = \frac{a - b}{a}$ (the ellipticity), $\frac{b^2}{a^2} = 1 - 2\epsilon$ nearly, and the mass

$$= 2\pi\epsilon\rho \, . \, \frac{a^2}{b^2} (2bx - x^2) \, \delta x,$$

$$= 2\pi\epsilon\rho (2bx - x^2) \, \delta x, \text{ nearly.}$$

Now every particle of this annulus may be considered at the same distance $\sqrt{2bx}$ from the attracted point; and, therefore, its attraction in the direction of the minor axis

$$= \frac{2\pi\epsilon\rho \, (2bx - x^2) \, \delta x}{2bx} \, . \, \frac{x}{\sqrt{2bx}},$$

$$= \frac{\pi\epsilon\rho}{b\sqrt{2b}} \, . \, (2bx^{\frac{1}{2}} - x^{\frac{3}{2}}) \, \delta x;$$

and therefore the attraction of the part which is exterior to the sphere

$$= \frac{2\pi\epsilon\rho}{b\sqrt{2b}} \, . \left(\frac{2}{3} b x^{\frac{3}{2}} - \frac{1}{5} \, . \, x^{\frac{5}{2}}\right) + C,$$

$$= \tfrac{16}{15} \, \pi\epsilon\rho b, \text{ between } x = 0, 2b.$$

But the attraction of the sphere $= \frac{4}{3} \, \pi\rho b$,

and ∴ the attraction of the spheroid

$$= \tfrac{4}{3} \, \pi\rho b \, (1 + \tfrac{4}{5} \, \epsilon).$$

25. Cor. If b be greater than a, the spheroid will be prolate, and ϵ will be negative; and therefore the attraction of a prolate spheroid on a particle at its pole

$$= \tfrac{4}{3} \, \pi\rho b \, (1 - \tfrac{4}{5} \, \epsilon).$$

26. **To find the attraction of an oblate spheroid of uniform density and small ellipticity on a particle at its equator.**

Let the particle be taken for the origin and the diameter of the equatorial circle passing through it for the axis of x; and circumscribe a sphere about the spheroid.

Then the equation of the generating ellipse is

$$y^2 = \frac{b^2}{a^2}(2ax - x^2),$$

and $\therefore$ the mass of the portion between the spheroid and sphere of a slice comprehended between two planes, at right $\angle^s$ to the axis of x, respectively at the distances x, and $x + \delta x$ from the attracted point,

$$= \pi\left(\frac{a^2}{b^2} - \frac{a}{b}\right)\rho y^2 \delta x = \pi\rho\epsilon(2ax - x^2)\delta x, \text{ ultimately};$$

and every particle of this mass may be considered at the same distance $\sqrt{2ax}$ from the attracted point; and therefore its attraction in the direction of the axis of x

$$= \pi\rho\epsilon . \frac{2ax - x^2}{2ax} . \frac{x}{\sqrt{2ax}} . \delta x,$$

$$= \frac{\pi\rho\epsilon}{2a\sqrt{2a}} . (2ax^{\frac{1}{2}} - x^{\frac{3}{2}}) . \delta x;$$

and therefore the attraction of the part which is exterior to the spheroid

$$= \frac{\pi\rho\epsilon}{a\sqrt{2a}} . \left(\frac{2}{3}ax^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}}\right) + \text{C},$$

$$= \tfrac{8}{15}\pi\epsilon\rho a, \text{ between } x = 0, 2a;$$

and since the attraction of the sphere $= \frac{4}{3}\pi\rho a$,

$\therefore$ the attraction of the spheroid $= \frac{4}{3}\pi\rho a\left(1 - \frac{2}{5}\epsilon\right)$.

27. Cor. If b be greater than a, the spheroid will be prolate, and its attraction on a particle at its equator

$$= \tfrac{4}{3}\pi\rho a\left(1 + \tfrac{2}{5}\epsilon\right).$$

28. To find the attraction of an oblate spheroid of uniform density on a particle at its pole.

Taking the attracted point as origin, and putting r for the distance of a small element from that point, and θ the $\angle$ which r makes with the axis of revolution, a small element of the generating ellipse $= r\delta r\delta\theta$, and the mass generated by the revolution of this element

$$= 2\pi\rho r^2\delta r \,.\, \sin\theta \,.\, \delta\theta\,;$$

and the attraction of this annulus in the direction of the axis of x

$$= 2\pi\rho\delta r \,.\, \sin\theta\cos\theta \,.\, \delta\theta\,;$$

and therefore the attraction of the spheroid

$$= 2\pi\rho \,.\! \int_r\!\int_\theta \sin\theta\cos\theta,$$

$$= 2\pi\rho \,.\! \int_\theta (r + \mathrm{C})\sin\theta\cos\theta\,;$$

the integral is to be taken from $r = 0$, to $r = \dfrac{2ba^2\cos\theta}{b^2 + a^2e^2\cos^2\theta}$; this value of r being determined from the equation of the generating ellipse $y^2 = \dfrac{a^2}{b^2}\cdot(2bx - x^2)$, by writing $r\sin\theta$ and $r\cos\theta$ respectively for y and x, and e for the eccentricity.

Hence the attraction

$$= \frac{2\pi\rho}{e^2} \,.\! \int_\theta \frac{2ba^2e^2\cos^2\theta\sin\theta}{b^2 + a^2e^2\cos^2\theta},$$

$$= \frac{4b\pi\rho}{e^2} \,.\! \int_\theta \left\{\sin\theta - \frac{b^2\sin\theta}{b^2 + a^2e^2\cos^2\theta}\right\} \quad \ldots\ldots (1),$$

$$= \frac{4b\pi\rho}{e^2} \,.\, \left\{\mathrm{C} - \cos\theta + \frac{b}{ae} \,.\, \tan^{-1}\left(\frac{ae\cos\theta}{b}\right)\right\},$$

$$= \frac{4b\pi\rho}{e^2} \,.\, \left(1 - \frac{b}{ae} \,.\, \tan^{-1}\frac{ae}{b}\right),$$

$$= 4b\pi\rho\left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3} \,.\, \sin^{-1}e\right);$$

the integral being taken from $\theta = 0$, to $\theta = \frac{\pi}{2}$, for the whole spheroid.

29. Cor. If the spheroid be prolate, b will be greater than a, and e^2 will be negative; and, therefore, equation (1) becomes

the attraction of a prolate spheroid

$$= \frac{4b\pi\rho}{e^2} \cdot \int_\theta \left\{ \frac{b^2 \sin\theta}{b^2 - a^2 e^2 \cos^2\theta} - \sin\theta, \right\},$$

$$= \frac{4b\pi\rho}{e^2} \cdot \left\{ \frac{b}{ae} \log_e \left(\frac{b - ae\cos\theta}{b + ae\cos\theta} \right)^{\frac{1}{2}} + \cos\theta \right\} + \mathrm{C},$$

$$= \frac{4b\pi\rho}{e^2} \cdot \left\{ \frac{b}{ae} \cdot \log_e \left(\frac{b + ae}{b - ae} \right)^{\frac{1}{2}} - 1 \right\},$$

$$= 4b\pi\rho \cdot \left\{ \frac{\sqrt{1 + e^2}}{e^3} \cdot \log_e \left(\frac{\sqrt{1 + e^2} + e}{\sqrt{1 + e^2} - e} \right) - \frac{1}{e^2} \right\}.$$

30. To find the attraction of an oblate spheroid of uniform density on a particle at its equator.

Let the particle be taken for the origin; the equator for the plane xy; and that diameter of it which passes through the particle for the axis of x; r the distance of an element of the spheroid from the origin; θ the inclination of r to the plane xy; ϕ the inclination of its projection on that plane to the axis of x.

Then an element of the mass $= \rho r^2 \delta r \,.\, \cos\theta \,.\, \delta\theta \,.\, \delta\phi$;
and the attraction of this element in direction of the axis of x, (which is evidently the direction in which the whole attraction of the spheroid takes place)

$$= \rho\delta r \,.\, \cos\theta \,.\, \delta\theta\, \delta\phi \,.\, \cos\theta\cos\phi;$$

and, therefore, the attraction of the spheroid

$$= \rho \cdot \int_r \int_\theta \int_\phi \cos^2\theta \cos\phi,$$

$$= \rho \cdot \int_\theta \int_\phi (r + \mathrm{C}) \cos^2\theta \cos\phi,$$

$$= \frac{2\rho b^2}{ae^2} \cdot \int_\theta \int_\phi \frac{e^2 \cos^3\theta \cos^2\phi}{1 - e^2\cos^2\theta},$$

(the integral being taken from $r = 0$, to $r = \dfrac{2b^2 \cos\theta\cos\phi}{a(1 - e^2\cos^2\theta)}$; this value of r being deduced from the equation of the spheroidal surface $x^2 + y^2 + \frac{a^2 z^2}{b^2} = 2ax$, by writing $r\cos\theta$ $\cos\phi$, $r\cos\theta\sin\phi$, and $r\sin\theta$ respectively for x, y and z),

$$= \frac{2\rho b^2}{ae^2}\int_\theta\int_\phi \cdot \cos^2\phi \left\{\frac{\cos\theta}{1-e^2+e^2\sin^2\theta} - \cos\theta\right\},$$

$$= \frac{2\rho b^2}{ae^2}\cdot\int_\theta\int_\phi\cdot\cos^2\phi\left\{\frac{a^2\cos\theta}{b^2+a^2e^2\sin^2\theta} - \cos\theta\right\} \ldots (1),$$

$$= \frac{2\rho b^2}{ae^2}\cdot\int_\phi\cdot\cos^2\phi\left\{\frac{a}{be}\cdot\tan^{-1}\frac{ae\sin\theta}{b} - \sin\theta + C\right\},$$

$$= \frac{2\rho b^2}{ae^2}\cdot\int_\phi (1+\cos 2\phi)\left(\frac{a}{be}\cdot\sin^{-1}e - 1\right),$$

(the integral being taken between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$),

$$= \frac{2\rho b^2}{ae^2}\left(\phi + \tfrac{1}{2}\sin 2\phi\right)\left(\frac{a}{be}\cdot\sin^{-1}e - 1\right) + C,$$

$$= 2\pi\rho b.\left(\frac{\sin^{-1}e}{e^3} - \frac{\sqrt{1-e^2}}{e^2}\right),$$

the integral being taken between $\phi = -\frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$.

31. COR. When b is greater than a, the spheroid is prolate, and e^2 is negative; and, from equation (1), we have the attraction of a prolate spheroid on a particle at its equator

$$= \frac{2\rho b^2}{ae^2}\cdot\int_\theta\int_\phi \cos^2\phi\left\{\cos\theta - \frac{a^2\cos\theta}{b^2-a^2e^2\sin^2\theta}\right\},$$

$$= \frac{2\rho b^2}{ae^2}\cdot\int_\phi\cos^2\phi\left\{\sin\theta - \frac{a}{be}\cdot\log_e\left(\frac{b+ae\sin\theta}{b-ae\sin\theta}\right)^{\frac{1}{2}}\right\} + C,$$

$$= \frac{2\rho b^2}{ae^2}\cdot\int_\phi(1+\cos 2\phi)\left\{1 - \frac{a}{be}\cdot\log_e\left(\frac{b+ae}{b-ae}\right)^{\frac{1}{2}}\right\},$$

$$= \frac{2\rho b^2}{ae^2}\cdot\left(\phi+\tfrac{1}{2}\sin 2\phi\right)\left\{1-\frac{a}{be}\cdot\log_e\left(\frac{b+ae}{b-ae}\right)^{\frac{1}{2}}\right\},$$

$$= 2\pi\rho b.\left(\frac{\sqrt{1+e^2}}{e^2} - \frac{1}{e^3}\cdot\log_e\left(\frac{\sqrt{1+e^2}+e}{\sqrt{1+e^2}-e}\right)\right).$$

32. If V be the sum of the quotients of the particles of a body divided by their distances from a given point, whose co-ordinates are f, g, h; then the attractions F, G, H of the body on a particle situated at that point in the directions of the co-ordinates, are respectively equal to

$$-d_f\mathrm{V}, \quad -d_g\mathrm{V}, \quad -d_h\mathrm{V};$$

and V may be determined from the equation,

$$d_f^2\mathrm{V} + d_g^2\mathrm{V} + d_h^2\mathrm{V} = 0;$$

$$\text{or, } d_f^2\mathrm{V} + d_g^2\mathrm{V} + d_h^2\mathrm{V} = -4\pi\rho;$$

the latter only being true when the particle attracted is a part of the body.

For, let $x\,y\,z$ be the co-ordinates of an element m of the body, and $u =$ its distance from the attracted point; then

$$u^2 = (f-x)^2 + (g-y)^2 + (h-z)^2$$

$$\text{and } \mathrm{V} = \Sigma\left(\frac{m}{u}\right),$$

$$\text{and } \therefore d_f\mathrm{V} = \Sigma\left(d_f\frac{m}{u}\right) = -\Sigma\left(\frac{m}{u^2}.d_fu\right)$$

$$= -\Sigma\left(\frac{m}{u^2}.\frac{f-x}{u}\right) = -\mathrm{F}.$$

Similarly, $d_g\mathrm{V} = -\mathrm{G}$, $d_h\mathrm{V} = -\mathrm{H}$.

Again, $ud_fu = f - x$;

and, differentiating again,

$$ud_f^2u + d_fu^2 = 1;$$

$$\therefore d_f^2u = \frac{1}{u} - \frac{d_fu^2}{u}.$$

$$\text{Now } d_f^2\mathrm{V} = -d_f\Sigma\left(\frac{m}{u^2}.d_fu\right),$$

$$= \Sigma\left(\frac{2m}{u^3}.d_fu^2 - \frac{m}{u^2}.d_f^2u\right),$$

$$= \Sigma\left(\frac{3m}{u^3}.d_fu^2 - \frac{m}{u^3}\right).$$

$$\text{Similarly, } d_g^2\mathrm{V} = \Sigma\left(\frac{3m}{u^3}.d_gu^2 - \frac{m}{u^3}\right),$$

$$\text{and } d_h^2\mathrm{V} = \Sigma\left(\frac{3m}{u^3}.d_hu^2 - \frac{m}{u^3}\right);$$

$$\therefore\; d_f^2\mathrm{V} + d_g^2\mathrm{V} + d_h^2\mathrm{V} = 0, \;.\;.\;.\; (1),$$

because

$$d_f u^2 + d_g u^2 + d_h u^2 = \left(\frac{f-x}{u}\right)^2 + \left(\frac{g-y}{u}\right)^2 + \left(\frac{h-z}{u}\right)^2 = 1.$$

33. Equation (1) is universally true when the particle attracted is not a part of the attracting body; it fails, however, in the contrary case, for then u becomes evanescent between the limits of the sum of $\left(\frac{m}{u}\right)$ for the whole body. To determine the form of equation (1), in this case, let a spherical surface having its centre in the origin of co-ordinates be inscribed within the body so as to include the particle attracted within the mass of the sphere thus formed; the function V will thus be separated into two parts U, U′; of which U belongs to the sphere, and U′ to the part of the body exterior to the sphere; since, therefore, the attracted particle is not a portion of this latter body, we have, by the former part of this article,

$$d_f^2\mathrm{U}' + d_g^2\mathrm{U}' + d_h^2\mathrm{U}' = 0;$$

and, therefore,

$$d_f^2\mathrm{V} + d_g^2\mathrm{V} + d_h^2\mathrm{V} = d_f^2\mathrm{U} + d_g^2\mathrm{U} + d_h^2\mathrm{U}.$$

But the three quantities $-\,d_f\mathrm{U}$, $-\,d_g\mathrm{U}$, $-\,d_h\mathrm{U}$, being the attractions which the sphere exercises on the particle in the directions of the co-ordinates, will be respectively equal to $\frac{4}{3}\pi\rho f$, $\frac{4}{3}\pi\rho g$, $\frac{4}{3}\pi\rho h$, by Art. 15; that is,

$$d_f\mathrm{U} = -\,\tfrac{4}{3}\pi\rho f,\quad d_g\mathrm{U} = -\,\tfrac{4}{3}\pi\rho g,\quad d_h\mathrm{U} = -\,\tfrac{4}{3}\pi\rho h.$$

Differentiating each of these equations, we have

$$d_f^2\mathrm{U} = -\,\tfrac{4}{3}\pi\rho,\quad d_g^2\mathrm{U} = -\,\tfrac{4}{3}\pi\rho,\quad d_h^2\mathrm{U} = -\,\tfrac{4}{3}\pi\rho;$$

and, therefore,

$$d_f^2\mathrm{V} + d_g^2\mathrm{V} + d_h^2\mathrm{V} = -\,4\pi\rho \;.\;.\;.\;.\;.\;.\; (2).$$

34. As an example of the application of equation (1), suppose it were required to find the attraction of a sphere on an external particle.

Let the centre of the sphere be the origin of co-ordinates, and $b^2 = f^2 + g^2 + h^2$. Then, from the symmetrical form of a sphere, so long as b remains the same, the attraction is evidently the same; and $\therefore$ V is a function of b only; consequently

$$d_f V = d_b V . d_f b = d_b V . \frac{f}{b},$$

$$\therefore d_f^2 V = d_b \left(d_b V . \frac{f}{b}\right) . d_f b = d_b^2 V . \frac{f^2}{b^2} + \frac{1}{b} . d_b V - d_b V . \frac{f^2}{b^3}.$$

$$\text{Similarly, } d_g^2 V = d_b^2 V . \frac{g^2}{b^2} + \frac{1}{b} . d_b V - d_b V . \frac{g^2}{b^3},$$

$$\text{and } d_h^2 V = d_b^2 V . \frac{h^2}{b^2} + \frac{1}{b} . d_b V - d_b V . \frac{h^2}{b^3};$$

$$\therefore 0 = d_f^2 V + d_g^2 V + d_h^2 V = d_b^2 V + \frac{2}{b} . d_b V.$$

Multiplying by b^2, integrating once, and then dividing by b^2, we find the attraction of the sphere

$$= - d_b V = \frac{A}{b^2}.$$

35. As an application of equation (2), suppose the particle to be within the sphere.

$$\text{Then } - 4\pi\rho = d_b^2 V + \frac{2}{b} d_b V.$$

Multiplying by b^2, and integrating, we have

$$A - \tfrac{4}{3}\pi\rho b^3 = b^2 d_b V.$$

But when $b = 0$, $d_b V = 0$, and $\therefore$ $A = 0$; and, consequently, the attraction of a sphere on a point within it $= - d_b V = \frac{4}{3}\pi\rho b$.

36. Again, suppose it were required to find the attraction of a cylinder of indefinite length on an external particle.

It is evident, from the symmetrical form of the cylinder, and from its being of indefinite length, that the attraction will remain the same so long as b, the distance from its axis, is the same. Now, supposing the axis of the cylinder to be the axis of z, $b^2 = f^2 + g^2$; and V is a function of b only,

$$\therefore\ d_f^2V + d_g^2V = 0.$$

$$\text{But } d_f^2V = d_b^2V \,.\, \frac{f^2}{b^2} + \frac{1}{b} \,.\, d_bV - d_bV \,.\, \frac{f^2}{b^3},$$

$$\text{and } d_g^2V = d_b^2V \,.\, \frac{g^2}{b^2} + \frac{1}{b} \,.\, d_bV - d_bV \,.\, \frac{g^2}{b^3};$$

$$\therefore\ 0 = d_f^2V + d_g^2V = d_b^2V + \frac{1}{b} \,.\, d_bV.$$

Multiplying by b, and integrating, we have the attraction =

$$-d_bV = \frac{A}{b}.$$

Consequently the attraction of a cylinder of indefinite length varies inversely as the distance from its axis.

37. If the particle be within the cylinder.

$$\text{Then } -4\pi\rho = d_b^2V + \frac{1}{b} \,.\, d_bV,$$

$$\therefore\ A - 2\pi\rho b^2 = b \,.\, d_bV.$$

But when $b = 0$, $d_bV = 0$, and $\therefore$ A $= 0$.

Hence the attraction of an indefinite cylinder on one of its own particles

$$= -d_bV = 2\pi\rho b.$$

38. Lastly, suppose it were required to find the attraction of a solid, of indefinite extent, on an external particle; the solid being bounded on the side next to the particle by a plane.

Here V is a function of b (the distance from the surface) only; and $b = f$,

$$\therefore\ d_b^2V = 0;$$

Consequently, the attraction ($-d_bV$) is constant.

ON THE ATTRACTION OF AN ELLIPSOID.

39. Let the co-ordinate axes of x, y, z, coincide with the semi-axes a, b, c, of the ellipsoid; let f, g, h, be the co-ordinates of the attracted particle, which take for the origin of polar co-ordinates; u the distance of an element m of the ellipsoid from this origin; θ the $\angle$ which u makes with the axis of x; and ϕ the $\angle$ which its projection on the plane yz makes with the axis of y. Then m may be considered as a small rectangular parallelopiped whose dimensions are δu, $u\,\delta\theta$, $u \sin\theta \,.\, \delta\phi$; and

$$\therefore m = u^2 \sin\theta \,.\, \delta u \; \delta\theta \; \delta\phi,$$

considering the density uniform, and equal to 1.

Consequently,

$$\mathrm{F} = \Sigma\left(\frac{m}{u^2} \cdot \cos\theta\right) = \int_u \int_\theta \int_\phi \sin\theta \cos\theta.$$

Similarly,

$$\mathrm{G} = \int_u \int_\theta \int_\phi \sin^2\theta \cos\phi, \text{ and } \mathrm{H} = \int_u \int_\theta \int_\phi \sin^2\theta \sin\phi.$$

In integrating, with respect to u, we observe that two cases occur, according as the particle is *within* or *without* the ellipsoid.

40. In the former case, the line which passes through the particle attracted, is divided into two parts u, u', which are terminated by the surface of the ellipsoid; the integrals are therefore to be taken between these limits, and consequently

$$\mathrm{F} = \int_\theta \int_\phi (u' + u) \sin\theta \cos\theta,$$
$$\mathrm{G} = \int_\theta \int_\phi (u' + u) \sin^2\theta \cos\phi,$$
$$\mathrm{H} = \int_\theta \int_\phi (u' + u) \sin^2\theta \sin\phi.$$

These expressions are now to be successively integrated, with regard to θ and ϕ, from θ and ϕ each $= 0$, to each $\doteq \pi$.

41. In the latter case, u cuts the ellipsoid in two points, the distances of which from the particle attracted are u and u'; the integrals are therefore to be taken between these limits; consequently

$$\begin{aligned} F &= \int_\theta \int_\phi (u' - u) \sin\theta \cos\theta, \\ G &= \int_\theta \int_\phi (u' - u) \sin^2\theta \cos\phi, \\ H &= \int_\theta \int_\phi (u' - u) \sin^2\theta \sin\phi. \end{aligned}$$

These expressions are now to be integrated between the values of θ and ϕ, corresponding to the points where $u' - u = 0$, that is, where u is a tangent to the ellipsoid.

42. Now $x = f - u\cos\theta$, $y = g - u\sin\theta\cos\phi$, $z = h - u\sin\theta\sin\phi$; and substituting these in the equation of the surface of the ellipsoid, in order to determine the values of u and u', we have for that purpose the equation

$$\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2} + \frac{\sin^2\theta\sin^2\phi}{c^2}\right).u^2 -$$
$$\left(\frac{f\cos\theta}{a^2} + \frac{g\sin\theta\cos\phi}{b^2} + \frac{h\sin\theta\sin\phi}{c^2}\right).2u$$
$$= 1 - \frac{f^2}{a^2} - \frac{g^2}{b^2} - \frac{h^2}{c^2};$$

from which, by writing L and M for the coefficients of u^2 and $2u$, and N for $M^2 + L.\left(1 - \frac{f^2}{a^2} - \frac{g^2}{b^2} - \frac{h^2}{c^2}\right)$, we obtain

$$u' = \frac{M + \sqrt{N}}{L}, \text{ and } u = \frac{M - \sqrt{N}}{L}.$$

Consequently, when the particle is *within* the ellipsoid,

$$\left.\begin{aligned} F &= 2 \int_\theta \int_\phi \left(\frac{M}{L}.\sin\theta\cos\theta\right), \\ G &= 2 \int_\theta \int_\phi \left(\frac{M}{L}.\sin^2\theta\cos\phi\right), \\ H &= 2 \int_\theta \int_\phi \left(\frac{M}{L}.\sin^2\theta\sin\phi\right), \end{aligned}\right\} \ldots\ldots (A),$$

and when the particle is *without* the ellipsoid,

$$F = 2 \int_\theta \int_\phi \left(\frac{\sqrt{N}}{L} . \sin\theta \cos\theta \right),$$

$$G = 2 \int_\theta \int_\phi \left(\frac{\sqrt{N}}{L} . \sin^2\theta \cos\phi \right),$$

$$H = 2 \int_\theta \int_\phi \left(\frac{\sqrt{N}}{L} . \sin^2\theta \sin\phi \right).$$

43. On account of the irrational quantity $\sqrt{N}$, it is quite impossible to integrate these latter expressions by any known method. We are able, however, to elude the difficulty, by making the attraction of the ellipsoid on an *external* particle depend upon the attraction of another ellipsoid on an *internal* particle, by a process, first discovered by Mr. Ivory, which the reader will find detailed in Airy's *Heterogeneous Figure of the Earth.* For this reason, we shall confine ourselves to the determination of the attraction of an ellipsoid on an *internal* particle.

44. For this purpose, substituting for M its value in the first of equations (A), we find,

$$F = \frac{2f}{a^2} . \int_\theta \int_\phi \frac{\sin\theta \cos^2\theta}{L} + \frac{2g}{b^2} . \int_\theta \int_\phi \frac{\sin^2\theta \cos\theta \cos\phi}{L} + \frac{2h}{c^2} . \int_\theta \int_\phi \frac{\sin^2\theta \cos\theta \sin\phi}{L}.$$

Now, it is well known, if P be any rational function whatever of $\sin\theta$ and $\cos^2\theta$, that $\int_{\theta = 0,\, \pi} P \cos\theta = 0$; hence the two last terms in the above expression disappear by integration; and, consequently, writing its value for L, we have

$$F = 2f . \int_\theta \int_\phi \frac{\sin\theta \cos^2\theta}{\cos^2\theta + \frac{a^2}{b^2} . \sin^2\theta \cos^2\phi + \frac{a^2}{c^2} . \sin^2\theta \sin^2\phi}.$$

Similarly,

$$G = 2g . \int_\theta \int_\phi \frac{\sin^3\theta \cos^2\phi}{\sin^2\theta \cos^2\phi + \frac{b^2}{a^2} . \cos^2\theta + \frac{b^2}{c^2} . \sin^2\theta \sin^2\phi},$$

and

$$\mathrm{H} = 2h . \int_\theta \int_\phi \frac{\sin^3\theta \sin^2\phi}{\sin^2\theta \sin^2\phi + \frac{c^2}{a^2} . \cos^2\theta + \frac{c^2}{b^2} . \sin^2\theta \cos^2\phi}.$$

45. Dividing these three equations respectively by f, g and h, and adding the results, we obtain

$$\frac{\mathrm{F}}{f} + \frac{\mathrm{G}}{g} + \frac{\mathrm{H}}{h} = 2 \int_\theta \int_\phi \sin\theta = 4\pi,$$

a most remarkable equation.

46. The values of F, G and H, contain only the ratios of the quantities a, b, c; so long, therefore, as these ratios remain the same, F, G, H will remain unaltered; whence we conclude that all similar ellipsoids exert the same attraction on an internal particle whose co-ordinates are given.

47. And from this it follows, that an ellipsoidal shell, contained between similar concentric ellipsoidal surfaces, exerts no attraction on a particle within it; because it is the difference between two similar and concentric ellipsoids, which, by last Art., exert equal attractions on the particle.

48. The integrals in the last expressions for F, G and H, are to be taken between limits (see Art. 40) which are independent of f, g and h, and therefore the integrals themselves are independent of these quantities; consequently

$$\mathrm{F} \propto f, \quad \mathrm{G} \propto g, \quad \mathrm{H} \propto h.$$

49. Hence F remains the same so long as f is unaltered; let, therefore, another particle be placed in such a situation that g and h may vanish, but f remain the same as before;

then, since the attraction of the ellipsoidal shell, which is contained between the external surface, and a similar concentric surface passing through the second attracted particle, is evanescent (Art. 47), we conclude that the attraction of the whole ellipsoid on the first particle is equal to the attraction of the small ellipsoid, contained by the second surface, on the second particle. (See Airy's Tracts, *Homogeneous Figure of the Earth*, prop. 8).

50. Integrating now the expression for F in Art. 44, with regard to ϕ, between the limits $\phi = 0$, and $\phi = \pi$, we find

$$F = 2f\pi . \int_\theta \frac{\sin\theta \cos^2\theta}{\left(\cos^2\theta + \frac{a^2}{b^2} . \sin^2\theta\right)^{\frac{1}{2}} . \left(\cos^2\theta + \frac{a^2}{c^2} . \sin^2\theta\right)^{\frac{1}{2}}} .$$

The integral of this is to be taken from $\theta = 0$, to $\theta = \pi$, which comes to the same as taking it from $\theta = 0$, to $\theta = \frac{\pi}{2}$, and doubling the result. Hence, by writing v for $\cos\theta$, and μ for the mass of the ellipsoid, we obtain

$$F = \frac{3\mu f}{a^3} . \int_v \frac{v^2}{\left\{1 + \left(\frac{b^2}{a^2} - 1\right)v^2\right\}^{\frac{1}{2}} . \left\{1 + \left(\frac{c^2}{a^2} - 1\right)v^2\right\}^{\frac{1}{2}}},$$

the integral of which is to be taken from $v = 0$, to $v = 1$.

51. If this integration could be effected, it is evident that v would disappear in the result; and therefore F is a function of f, a, b, c only; consequently G and H are *similar* functions of g, b, a, c; and h, c, a, b; respectively.

$$\therefore G = \frac{3\mu g}{b^3} . \int_v \frac{v^2}{\left\{1 + \left(\frac{a^2}{b^2} - 1\right)v^2\right\}^{\frac{1}{2}} . \left\{1 + \left(\frac{c^2}{b^2} - 1\right)v^2\right\}^{\frac{1}{2}}},$$

and

$$H = \frac{3\mu h}{c^3} . \int_v \frac{v^2}{\left\{1 + \left(\frac{b^2}{c^2} - 1\right)v^2\right\}^{\frac{1}{2}} . \left\{1 + \left(\frac{a^2}{c^2} - 1\right)v^2\right\}^{\frac{1}{2}}},$$

between the same limits of v as before.

52. It is impossible to exhibit the integral, with regard to v, in finite terms; we may, however, put the values of F, G and H under a somewhat more convenient form, as follows.

For $\frac{b^2}{a^2} - 1$, and $\frac{c^2}{a^2} - 1$, substitute e^2 and ϵ^2 respectively; and in the value of G write $\frac{bv'}{a\sqrt{1 + e^2 v'^2}}$ for v; and in that of H write $\frac{cv''}{a\sqrt{1 + \epsilon^2 v''^2}}$ for v; then

$$\mathrm{F} = \frac{3\mu f}{a^3} \cdot \int_v \frac{v^2}{\sqrt{1 + e^2 v^2} \cdot \sqrt{1 + \epsilon^2 v^2}},$$

$$\mathrm{G} = \frac{3\mu g}{a^3} \cdot \int_{v'} \frac{v'^2}{(1 + e^2 v'^2)^{\frac{3}{2}} \cdot (1 + \epsilon^2 v'^2)^{\frac{1}{2}}},$$

$$\mathrm{H} = \frac{3\mu h}{a^3} \cdot \int_{v''} \frac{v''^2}{\sqrt{1 + e^2 v''^2} \cdot (1 + \epsilon^2 v''^2)^{\frac{3}{2}}}.$$

Since, however, these integrals are all to be taken between $v = 0$ and $v = 1$, and at these limits v' and v'' have the same values as v, we may in the expressions for G and H write v for v' and v'' without altering the final results. This being done, and substituting

$$\mathrm{F}' \text{ for } \int_{v=0,1} \frac{v^2}{\sqrt{1 + e^2 v^2} \cdot \sqrt{1 + \epsilon^2 v^2}},$$

we have

$$\mathrm{F} = \frac{3\mu f}{a^3} \cdot \mathrm{F}', \quad \mathrm{G} = \frac{3\mu g}{a^3} \cdot d_e (e\,\mathrm{F}'), \quad \mathrm{H} = \frac{3\mu h}{a^3} \cdot d_\epsilon (\epsilon\,\mathrm{F}').$$

53. We can find F′ in finite terms only in two cases, when $e = \epsilon$, and when one of them $= 0$; on both which suppositions two of the three semi-axes a, b, c are equal; and, consequently, the ellipsoid becomes a spheroid of revolution, the attraction of which may be found independently of F′, by Arts. 28 . . 31. 49.

54. For a more extended view of the subject of the Attractions of Spheroids, and for the application of the results above deduced to the determination of the Figures of the Earth, and the heavenly bodies, the reader is referred to the Writings of Newton and Laplace; or to Airy's Mathematical Tracts, and the Théorie Analytique du Système du Monde, tome ii. of Pontécoulant.

PRINTED BY W. METCALFE, CAMBRIDGE.

AN INDEX

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ERRATA.

Page	*line*	
6.	15.	*For* cos 3, *read* cos 2.
38.	6.	*For* $d_p r$, *read* $d_r p$.
68.		Remove "∴ the pressure" from line 19 to the line above.
106.	15.	*For* $+\cos\theta$, *read* $-\cos\theta$.
107.	16.	*For* C, *read* h.
126.	2, of Ex. 1.	*For* Then, *read* The.
142.	2, of Cor. 1.	*For* $\log_e ht$, *read* $\log_e \sin ht$.
142.	4, of Cor. 1.	*For* $s =$ &c. *read* $s = 2\beta k \, . \sqrt{ag} \, . \, e^{-\frac{kg}{h} . \tan^{-1} \frac{h}{kg}}$.

The reader will observe that in Art. 152, m is supposed so small that its square may be neglected.

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